Institute of Actuaries of Australia

# Using the market implied risk aversion to value all risk

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## Abstract

This paper arose from a desire to include some sort of allowance for non-financial risk in a market consistent valuation. This desire led to a review of some literature that showed there were features of valuation techniques already used by actuaries that, if looked at in a slightly different way, could have a wide range of practical applications, including deriving the cost of any type and shape of risk. The purpose of this paper is to put down a broad description for deriving this cost of risk.

This paper describes an approach for valuing any set of future uncertain profits by looking at cost of risk already implicit in the value of financial assets traded in capital markets.

To be useful for enterprise risk management purposes and, arguably, published corporate value measures, the approach needs to include all sources of risk, not just financial risk. It needs to handle all shapes of profit distributions, both symmetrical and non-symmetrical, and it needs to value any financial risk consistently with observable market values for similar risks.

Overall, we want to derive a set of stochastic discount factors "m" such that, for any uncertain future outcome Z, the present value of Z = Σm.z.P(z), where P(z) is a real world distribution. In other words, value is real world expected value of mZ.

Stochastic discount factors have a number of applications and contain useful information:

- They explain the risk aversion inherent in the value of financial assets
- They can convert any uncertain future outcome into a risk adjusted value
- Under simple assumptions, they can derive a cost of capital and can recover the Black Scholes pricing formula

Not withstanding these strengths, there are limitations that need to be understood before these factors can be applied in all circumstances.


*Key words: risk aversion; stochastic discount factors; market consistent values; non-financial risk*

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# 1 Introduction

## 1.1 The desire

This paper arose from a desire to include some sort of allowance for non-financial risk in a market consistent valuation. This desire led to a review of a range of literature that showed there were features of valuation techniques already used by actuaries that, if looked at in a slightly different way, could have a wide range of practical applications, including deriving the cost of any type and shape of risk. The purpose of this paper is to put down a broad description for deriving this cost of risk.

The references gives examples of this literature and nearly all can be downloaded free of charge from the internet.

## 1.2 Purpose

This paper describes an approach for valuing any set of future uncertain profits by looking at cost of risk already implicit in the value of financial assets traded in the capital markets.

This could provide a useful benchmark for internal management purposes, where not all risk is diversifiable in the capital markets, and any residual risks can still cause the same practical issues as a financial risk. It may also help explain the difference between a pure market consistent value of a financial services company and its market capitalisation.

## 1.3 Structure

Section 2 sets out the goal for this paper. This goal is to find a general formula for finding the risk adjusted value of a set of future profits. To be useful for internal management purposes, this formula needs to consider risks from all sources.

A key constraint is that the approach needs to give a market consistent value when applied to financial risks. Therefore, a starting point is to look at whether a general formula can be applied for financial risks.

Section 3 explains market consistent values and how they allow for risk. This section shows that the process can be explained in terms of stochastic discount factors.

Section 4 explains how stochastic discount factors relate to utility theory and how they can be described in terms of risk aversion.

Section 5 describes what it would take to apply these stochastic discount factors to all risks.

Section 6 shows how stochastic discount factors can work for different profit distributions and how a risk adjusted distribution can be derived.

Section 7 briefly comments on some practical applications.

Section 8 outlines why the approach described in this paper may not work for all occasions.

The appendices contain additional, more technical information to support some of the analysis.

# 2 Goal

## 2.1 The goal

The goal of this paper is propose a general approach for deriving a risk adjusted value for any set of uncertain future profits.

The overall criteria for the approach are:

- it needs to produce a risk adjusted value; the higher the risk the lower the value (all other things being equal)
- it must be applicable for all shapes of profit distributions. It needs to handle both symmetrical and non-symmetrical risk
- it needs to be useful for enterprise risk management purposes. As a result, it needs to include all sources of risk
- the value of any financial risk needs to be consistent with observable market values for similar risks
- it needs to be explainable and comparable in fairly general terms

## 2.2 The formula

Suppose we have an uncertain future profit stream. Assume that we can derive a model for:

- Z, the future profit stream
- P(z), the probability that the profit will be some value z

Overall, we want to derive a set of factors "m" such that:

$$\boxed{\text{Value} = \Sigma m.z.P(z)}$$

That is, we want to know how we can adjust the range of possible future profits to get a risk adjusted value. In this formula, "m" is some set of a stochastic discount factors. Importantly, the stochastic discount factors may vary for different levels of profit.

This formula is the same as saying "value equals the expected value of mZ".

A starting point is looking at the constraint that the general formula needs to return a market consistent value for financial risks. This is described in the next section.

# 3 Financial risks and market consistent values

## 3.1 Market consistent values

Any general formula needs to return a market consistent value for financial risks.

Market consistent values (MCVs) may be calculated for a set of cashflows by finding the value of a market portfolio that replicates the cashflows under all market conditions.

This may be achieved by using risk neutral assumptions for investment returns. For more details on market consistent values, see:

- Jarvis, Southall and Varnell (2001), Modern Valuation Techniques
- Blight, Kapel and Bice (2003), Market Consistent Economic Valuations for Wealth Management Companies

## 3.2 Risk neutral approach for financial risks

This paper will take a simplified approach to describing the risk neutral approach for financial risks. In particular:

- the market contains only one asset class. For simplicity, we will assume that local equities are sufficient to explain the risk aversion of investors
- X is the future level of the market
- X is a continuous variable. This isn't strictly necessary but may make the formulae easier
- the risk free rate is constant and observable, and the risk free discount factor is v, where v = 1/(1+ the risk free rate)

Suppose that we wish to value a set of profits where:
Z = the range of future profits
V = value of the replicating portfolio for these profits
x = the future level of the market
$Z_x$ = the profit when the market is at x

MCVs work by finding some factor "Q" that we can apply to different future profits, then discount at the risk free rate. The risk adjusted value is then the sum of all these across all possible cashflows.

That is, they use some function Q(x) such that $V = \int Z_x vQ(x)dx$.

For this formula to work, vQ(x)dx needs to be the value of an asset that pays 1 when the market is between x and x + dx, and zero at all other times. Appendix 1 shows how such an asset can be constructed using options (in theory).

In 1978, Breeden and Litzenberger showed that vQ(x) can be derived from taking the second derivative of an option price curve with respect to the strike price. Appendix 1 goes on to show this is more detail.

Since it can be derived from the price of traded options, any two market practitioners should derive a similar for Q.

Q(x) is known as a risk neutral distribution as it is already risk adjusted. No further allowance for risk is required and any discounting can be at the risk free rate.

Black Scholes is a good example of a valuation function that uses a risk neutral distribution. Appendix 3 shows that the risk neutral distribution underlying Black Scholes for the future share price, $X_t$, is lognormal with $E(\ln(X_t)) = r - 0.5\sigma^2$, and the standard deviation of $\ln(X_t)$ is $\sigma$, where r is the risk free rate and $\sigma$ is the standard deviation of real world distribution of $\ln(X_t)$.

## 3.3 Real world distributions for financial risks

While we only need to know Q(x) to value a set of profits for a financial risk, we can still derive a set of stochastic discount factors to apply to a real world distribution.

Let P(x) be the real world distribution for the future level of the market. It measures the probability density function for the level of market.

Deriving P is not as straight forward as deriving Q as it cannot be directly based on currently observable market prices. P can vary from simple models to complex time series analyses, and it is unlikely that any two professionals will develop precisely the same parameters or even model. (The literature often refers to the real world probability as "subjective" as it depends upon the view of the modeller, and the risk neutral distribution as "objective" as it can be derived from market prices.)

Despite this complexity, this paper will assume that it is possible to model the market's view for the real world distribution P(x).

## 3.4 Stochastic discount factors

If we can derive a market view for P(x) then we can derive the stochastic discount factors for financial risks to apply to the real world distribution.

That is, if for a financial risk
V = the market consistent value of a set of cashflows
P = market's view of the real world distribution

We want to find some factor $m_x$ such that $V = \int Z_x m_x P(x)dx$

That is, $m_x$ is some weight to apply to each level of cashflow and get the average using the real world distribution.

Importantly, m is different for different future states of the market, and is linked through Q and P.

If $V = \int Z_x vQ(x)dx$ and $V = \int Z_x m_x P(x)dx$
then a solution is $vQ(x) = m_x P(x)$

So, for financial risks
$m_x$ = Stochastic discount factor $= \frac{Q(x)}{P(x)} v$, where "x" is the future level of market

This could be interpreted as "remove P and put in Q and discount at the risk free rate".

By itself, this is not necessarily that useful for valuing a financial risk. If we know Q, we should just use this function to value the profit in the first place.

However, $m_x$ is important in explaining how the risk is valued.

## 3.5 P, Q and the modern actuary

Actuaries are well placed for using P and Q.

P, the real world probability, has been used for many years and is used for enterprise risk management, including target surplus and risk based capital.

Q is used for the valuation and pricing of financial contracts, especially those with an asymmetric profit distribution that varies with the level of the market. Prime examples are put and call options, including options imbedded into financial services products.

The risk management of guaranteed products with implicit or explicit options use both P and Q. The P probability is used to get a range of possible outcomes, and Q is used to derive the value of the guarantee within each outcome. These are known as "stochastic within stochastic" projections.

Many actuaries may be using P and Q, but not be aware of the useful information that links them. In some cases, $m_x$ may be a neat equation if both P and Q have a similar functional form.

(The choice of P and Q for these functions is a little unfortunate for life insurance actuaries, who are used to p and q relating to mortality rates. These, however, have become the standard symbols across the literature.)

## 4 Stochastic discount factors, utility theory and risk aversion

### 4.1 Utility theory

To understand stochastic discount factors, it is worth going back to one of the cornerstones of economics, utility theory.

Utility theory is a well known concept, but it is worth covering some of the key principles.

Shuttleworth (1988) says utility theory "[attempts] to assign a relative value, or utility, to different levels of wealth". In simple terms, there is a function U(w) of wealth that measures utility where, for a risk adverse investor:

- $U'(x) > 0$
- $U''(x) < 0$

That is, utility increases with increasing wealth, but at a decreasing rate. (Put another way, the pain from losing a dollar is greater than the joy from gaining a dollar.)

Jarvis et al (2003) shows an important link between stochastic discount factors and utility theory. (Their paper refers to stochastic discount factors as "state price deflators", but this is a less common term.)

| For financial risks, stochastic discount factors are proportional to the marginal utility of the optimal market portfolio. |
|---|

This is an important relationship and provides the connection to a wider application of stochastic discount factors.

Many of the principles of utility theory can therefore be seen in the properties of the stochastic discount factors.

### 4.2 Examples of stochastic discount factors for financial markets

To see the properties of stochastic discount factors for financial markets, assume a simple functional form for P and Q

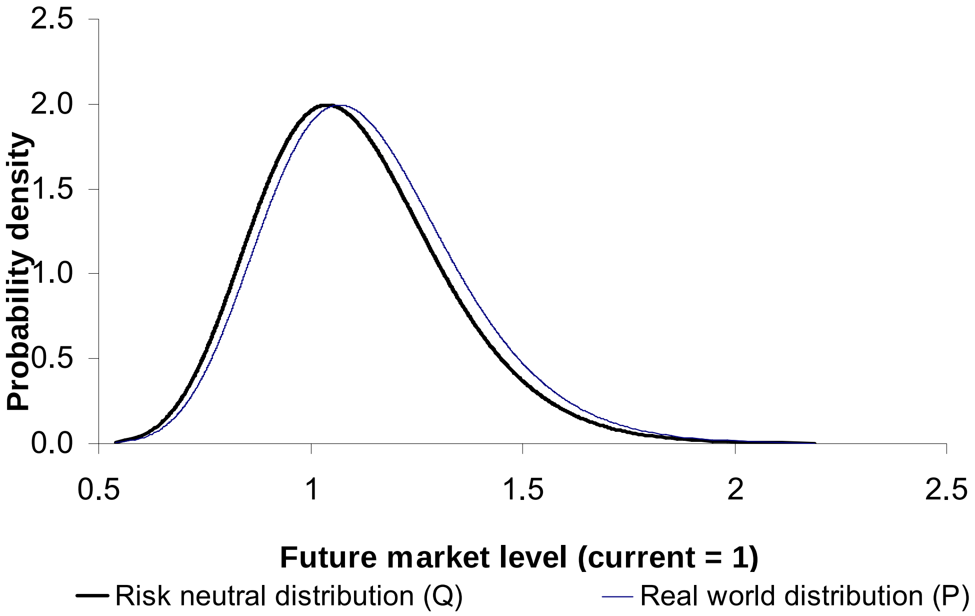
- Both P and Q are lognormal
- Risk free rate = 6% pa
- Expected return on shares = 9%
- Volatility = 20%

The following table shows the parameters for P and Q.

| Function | Type of distribution | μ | σ | Mean = $e^{\mu+\frac{\sigma^2}{2}}$ |
|---|---|---|---|---|
| P | Log normal | 6.618% | 20% | 1.09 |
| Q | Log normal | 3.827% | 20% | 1.06 |

The following graph shows each distribution.

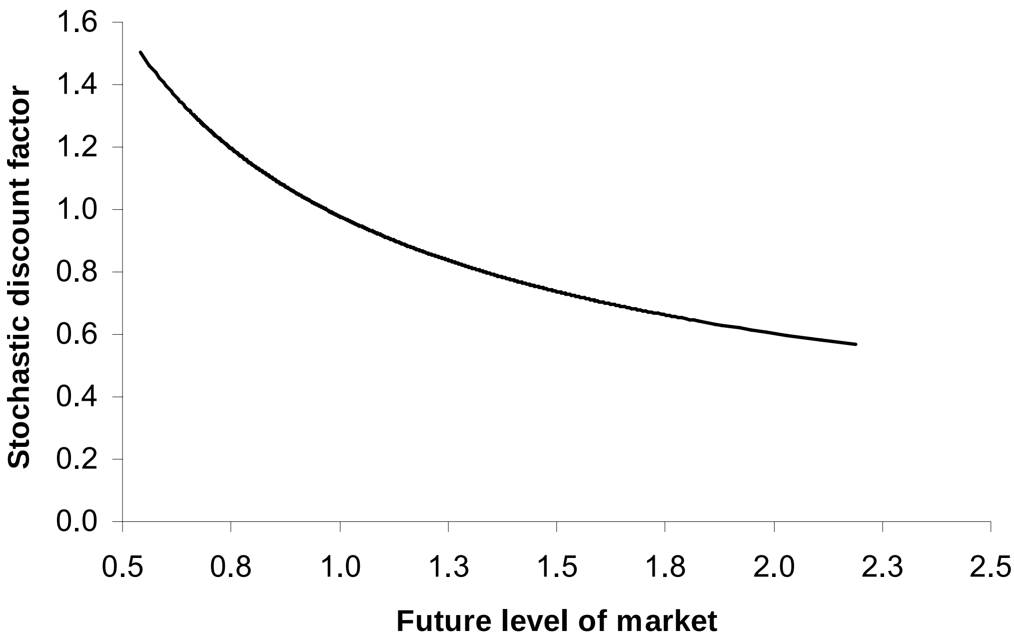
***Graph 4.1: Risk neutral (Q) and real world (P) distributions***




Note that the shape of Q and P is basically the same but Q has been shifted to the left. It gives more weight to the lower outcomes and less weight tho the higher ones. Although it is not obvious from the graph, the mean of the Q distribution is 1 + the risk free rate. The mean of the P distribution reflects the real world expected return and includes a margin for risk.

Graph 4.2 below shows the resulting stochastic discount factor. This is simply the ratio of Q over P, and then multiplied by v.

***Graph 4.2: Stochastic discount factors for different market levels (Current level =1)***



A formula can be derived for the above stochastic discount factors.

If,

- P is a lognormal distribution with parameters μ and σ
- Q is a lognormal distribution with parameters μ* and σ (the same sigma as P)

$$m_x = ve^{-[\lambda]\left[\frac{(\ln(x)-\mu)}{\sigma}+0.5\lambda\right]}$$

where $\lambda = \dfrac{(\mu - \mu^*)}{\sigma}$

A proof is provided in the appendix.

### 4.3 Properties of stochastic discount factors

If the market is risk adverse, the properties of stochastic discount factors are:

- They should always be positive but downwards sloping
- More weight is given when the market level is low
- Less weight is given when the market level is high
- The average value (using the real world probability) of $m_x$ is v.

The relationship with marginal utility helps explain the shape of stochastic discount factors. The stochastic discount factors give more weight to when the investor is desperate for money (in low markets) and less weight when the investor is flush with money (high scenarios).

This is still the same as the pain from losing a dollar is greater than the joy from gaining a dollar.

### 4.4 Risk aversion and utility functions

The key property of utility curves is not the absolute level but rather their shape. As mentioned above, the key requirements for utility functions are the first and second derivatives. A better measure of utility is to look at the curvature of the marginal utility, rather than the absolute level. (This would also mean that a good way of understanding stochastic discount factors is to look at their curvature.)

The most well known measures of this curvature for marginal utility were introduced by John Pratt (1964) and Kenneth Arrow (1965): absolute risk aversion and relative risk aversion.

Absolute risk aversion (ARA): $-\dfrac{U''(x)}{U'(x)}$ (or $= -\dfrac{m'(x)}{m(x)}$, since m is proportional to $U'$ )

Relative risk aversion (RRA): $-\dfrac{U''(x)}{U'(x)}x$ (or $= -\dfrac{m'(x)}{m(x)}x$)

More details on risk aversion are given in Appendix 2.

**Using the market implied risk aversion to value all risk**

In our simple model, $m(x) = ve^{-\lambda\left[\frac{(\ln(x)-\mu)}{\sigma}+0.5\lambda\right]}$

$$m'(x) = -\frac{\lambda}{\sigma x}.ve^{-\lambda\left[\frac{(\ln(x)-\mu)}{\sigma}+0.5\lambda\right]} = -\frac{\lambda}{\sigma x}.m(x)$$

So

Absolute risk aversion $= -\frac{m'(x)}{m(x)} = \frac{\lambda}{\sigma x}$

Relative risk aversion $= -\frac{m'(x)}{m(x)}.x = \frac{\lambda}{\sigma}$

Therefore, this simple model, which underpins Black Scholes, assumes constant relative risk aversion.

The $\lambda/\sigma$ ratio is often referred to as the co-efficient of relative risk aversion. $\lambda$ is the Sharpe ratio: the risk premium ($\mu$ - $\mu^*$) divided by the standard deviation $\sigma$. The co-efficient of relative risk aversion is therefore the risk premium divided by the variance.

In our example,

| | | |
|---|---|---|
| $\mu$ | 6.618% | |
| $\mu^*$ | 3.827% | |
| $\sigma$ | 20% | |
| $\lambda$ | 14.0% | $\lambda = \frac{(\mu-\mu^*)}{\sigma} = \frac{0.0662-0.0383}{0.2}$ |
| $\frac{\lambda}{\sigma}$ | 69.8% | $\frac{0.14}{0.2}$ or $\frac{0.0662 - 0.0383}{0.2^2}$ |

So, in this case, the co-efficient of relative risk aversion is 69.8%.

If there is constant relative risk aversion, the utility curve takes the form $U(x) = \xi\frac{1}{1-\frac{\lambda}{\sigma}}x^{1-\frac{\lambda}{\sigma}}$,

where $\xi$ is some constant.

Knowing “m” is helpful for explaining the risk aversion implied in the value of financial assets. The next section looks at whether we can apply this risk aversion more generally to all risks.

# 5 Applying the method to all risks

## 5.1 Market consistent values and non-financial risks

Non-financial risks can be defined as any risk that cannot be replicated using financial instruments. In lay terms, it is any risk that is not directly related to financial markets. It includes:

- some product related risks such as mortality risk and surrender risk
- operational risks such as losses from failures in processes, such as unit pricing errors

Unfortunately we can't apply a pure MCV approach to non-financial risks.

A pure MCV approach effectively assumes that non-financial risks are risk free as there is no portfolio that can replicate the cashflows. This is consistent with financial economic theory that says that since these risks are not correlated with financial markets, shareholders can reduce this cost to zero by investing in a well diversified portfolio of companies. (Many economists have demonstrated the practical difficulties in diversifying both financial and non-financial risk, such as limits in short selling.)

However, as outlined below, there are a number of reasons why there should be some allowance for non-financial risks in a useful value measure. Some financial services companies around the world have made attempts to include a margin for non-financial risk in their published MCVs, but there is no general agreement on a uniform approach.

## 5.2 Enterprise risk management

Only taking into account financial risk may limit the usefulness of pure MCVs for enterprise risk management (ERM).

The Casualty Actuarial Society defines ERM as "the process by which organizations in all industries assess, control, exploit, finance, and monitor risks from all sources for the purpose of increasing the organization's short and long term value to its stakeholders."

There are a number of points to make with this definition:

- risk is from all sources, not just financial risk
- the key purpose is to increase value. Any value measure should therefore look at risk from all sources.
- the value is to stakeholders and not just shareholders. Some stakeholders, such as management and regulators to name two, are interested in the impact from of all risks
- managing risk appropriately can add value to an enterprise

Therefore, as part of an ERM framework, it seems reasonable to include some allowance in the value for non-financial risk.

## 5.3 MCVs and market capitalisation

One criticism of pure MCVs is that they overstate market capitalisations for financial services companies. This is generally accepted in Australia and in other countries as well.

The difference between MCV and the market capitalisation is attributed to “frictional costs” or “agency costs”. Profit volatility from non-financial risks may be a direct or indirect driver of these costs. Reasons for this include:

- management unit costs may increase from the best estimate level if profits are low (e.g. one-off corporate restructuring costs) and also if profits are high (e.g. softer cost controls)
- higher profit volatility makes it more difficult to manage an entity. Strategic plans are more difficult to prepare, implement and achieve in an unstable environment. Therefore, any increase in profit volatility can make it more difficult to create value
- market analysts mark down company valuations if they cannot accurately forecast profit, even if the volatility is due to non-financial risks
- the market capitalisation of a company typically falls if dividends fall, even if the fall in the dividend is a result of non-financial risk
- companies need to hold capital for non-financial risks and need to raise additional capital if non-financial risks cause solvency issues. It is reasonable that all capital comes at a cost
- management policies, goals and remuneration are typically based on company profit from all sources. Therefore management considers any company wide risk when making business decisions
- management do not consider any diversification benefits with other companies that shareholder may be invested in. In practice, management has limited diversification opportunities and still must consider any residual risks

Therefore, it seems reasonable to allow for non-financial risks in some way, either as part of an internal ERM framework or even as part of published enterprise value.

## 5.4 Deriving company specific stochastic discount factors

As we can’t construct a replicating portfolio for non-financial risks, and we can’t observe a nice neat risk neutral function for these risks, we need to apply some sort of risk aversion to their real world distribution.

The question becomes what form of risk aversion to use. One approach is to derive a company specific risk aversion function from the company risk preferences. In practice, this may be difficult.

An objective alternative is to use the risk aversion already implicit in the value of financial risks and assume this is the company’s risk aversion. Applying market implied risk aversion to all the company’s risks serves as a (somewhat) observable method that benchmarks the cost of risk against some external measure.

However, the company risk aversion still needs to relate to the company’s utility. For this to work, the company’s utility needs to be a function of the company’s profit (and not a function of the level of the market).

To have the same risk aversion as the market, the shape of the company’s utility curve needs to be the same the market’s utility curve but aligned with company profit.

## Using the market implied risk aversion to value all risk

We can achieve this through appropriately constructed stochastic discount factors. We would require stochastic discount factors that:

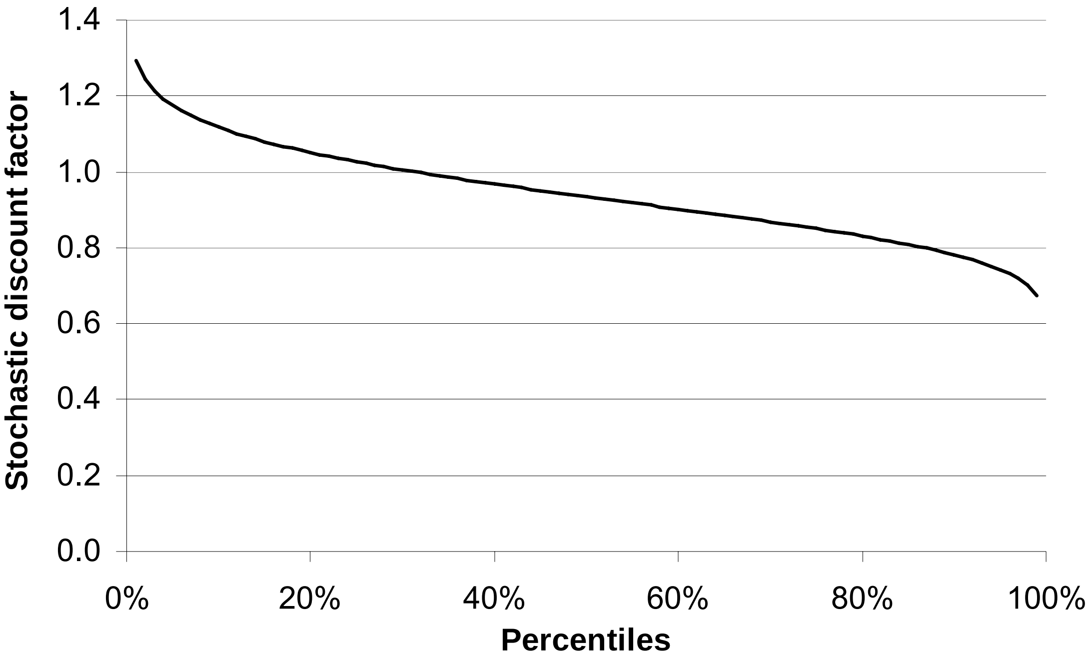
- are positive
- are highest when the profit is lowest
- are lowest when the profit is highest
- have the same shape as the stochastic discount factors for financial risks

To achieve this, we can use the same values for $m_x$ but just reorder then so that they are aligned with the driver of company utility. That is, they need to be aligned with company profit and not the level of the market.

As a first step, we need to convert the market based stochastic discount factors so that they relate to a percentile outcome, rather than the level of the market.

Graph 5.1 shows the stochastic discount factors for each percentile of the market, rather than the market level. These can then be used as the stochastic discount factors for the company profit percentiles. These factors in this graph are based on the simple model outlined in Section 4.2.

***Graph 5.1: Stochastic discount factors for different percentiles***




In this case, the function form is m(p) = $ve^{-[\lambda][C_p+0.5\lambda]}$
where

- p is a the percentile corresponding to a level of profit
- m(p) is the stochastic discount rate for the percentile p
- $C_p$ is the inverse standard normal variable with a probability of p, ( $C_p = \frac{\ln(x)-\mu}{\sigma}$ )

We can now apply these factors to a company profit to get a risk adjusted value. The next section shows that we can now transform a real world distribution to derive a risk adjusted value.

# 6 Impact of stochastic discount factors

## 6.1 A shift to the left

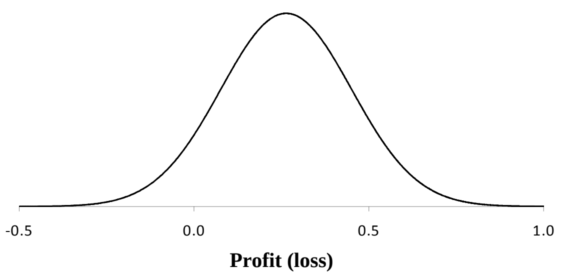
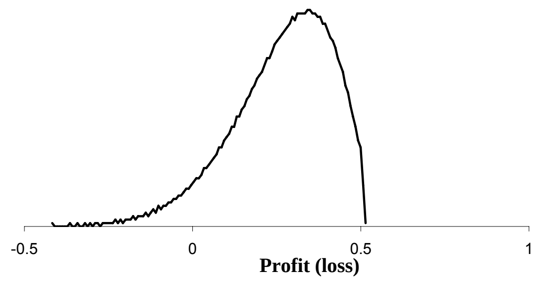
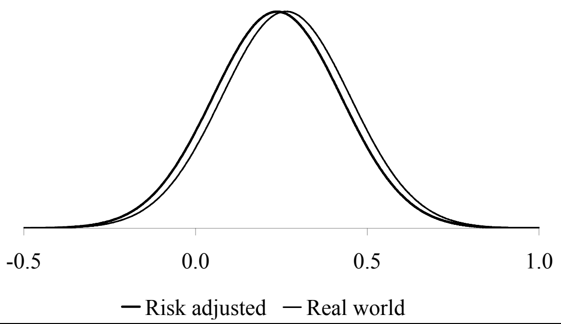
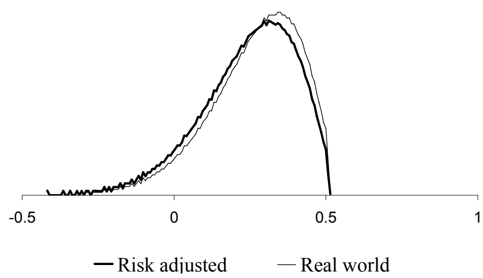
Assuming a positive risk aversion, stochastic discount factors effectively distort a profit distribution by giving more weight to the lower profits and less weight to the better outcomes.

Therefore, stochastic discount factors effectively shift a distribution to the left (provided profits are positive and losses are negative). This makes the risk adjusted value always less than mean by some margin (the "cost of risk"). The shift is larger when the difference between the downside outcomes and the upside outcomes is larger.

It can be helpful to look at what would be the risk adjusted variable before we discount it at the risk free rate. That is, what if we apply the Q/P ratio and do the discounting separately.

The graphs below show what happens to a variable if we apply these factors. Two examples are given: a normally distributed profit and a skewed profit. Note that in both cases, the distribution is shifted to the left. The mean of the risk adjusted distribution can then be discounted at the risk free rate to derive a risk adjusted present value.

***Graph 6.1: Examples of risk adjusted distributions***

<table>
<tr><th>Normal case</th><th>Asymmetric case</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Apply factors</td><td>Apply factors</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Mean<br>Real world 0.264<br>Risk adjusted 0.239</td><td>Mean<br>Real world 0.264<br>Risk adjusted 0.243</td></tr>
</table>

It is important to note that the new distribution cannot be used for understanding the likelihood of a particular event. They can only be used for value measures.

# 7 Applications

Stochastic discount factors and their effects on profit profiles have a number of applications. They can be applied to any uncertain future outcome to derive a risk adjusted value. (A key assumption, however, is there the robust model for the variable being valued.)

Some of the applications are touched on below.

## 7.1 Risk adjusted values

The approach can be used to determine a risk adjusted value for any risk profile. Furthermore, the cost of the risk is the consistent for financial and non-financial risks.

In simple formula terms, value = E(mZ) where Z is a random variable for the risk and m is the stochastic discount factors based on the risk aversion implied by the market's view of P and Q.

This value can then be used to derive a risk measure, namely the difference between the mean, discounted at the risk free rate, and the risk adjusted value.

In formula terms:

PV of cost of risk = vE(Z) – E(mZ)

where Z is a random variable for the risk and v is the risk free discount factor

There are a number of advantages in using this formula as a measure of the cost of risk. In particular:

- it can apply for any shaped distribution. Using standard deviation as a risk measure really only applies for normally distributed risks
- it takes into account the whole distribution, including both the upside and downside. Many risk measures such as VaR and Tail VaR only consider the downside
- it takes into account risk aversion, and the risk aversion does not need to take any particular form. Some risk measures, such as the "Wang Transform", effectively assume a constant relative risk aversion (see Wang 2000 and Wang 2002).

The following table shows the cost of risk for the examples in Section 6.1.

***Table 7.1: Risk adjusted values form normal and skewed examples***

| | Normal | Skewed | |
|---|---|---|---|
| Best estimate profit | 0.264 | 0.264 | a |
| *PV at risk free rate* | *0.249* | *0.249* | b = a / (1 + risk free rate) |
| Risk adjusted mean | 0.239 | 0.243 | c |
| *PV at risk free rate* | *0.225* | *0.230* | d = c / (1 + risk free rate) |
| Cost of risk | 0.025 | 0.021 | e = a -c |
| *PV of cost of risk* | *0.024* | *0.020* | f = d - b |

### 7.2 Risk adjusted values under simplified assumptions

Under simplified assumptions, stochastic discount factors can give a simple risk adjustment.

If $m_p = ve^{-\lambda[C_p+0.5\lambda]}$ then for a normally distributed random variable, the risk adjusted random variable is also normally distributed with the same standard deviation but the mean is shifted. The following table shows the new parameters.

If Z is normally distributed

| | **Real world** | **Risk adjusted** | **Comment** |
|---|---|---|---|
| Mean | $\alpha$ | $\alpha - \beta\lambda$ | Shifted by $\beta\lambda$ |
| Standard deviation | $\beta$ | $\beta$ | Unchanged |
| Where $\lambda = \frac{(\mu - \mu^*)}{\sigma}$, using the parameters for the lognormal distribution for P and Q. | | | |

A proof is given in the appendix.

In this case, there is no need to do a full projection to get the risk adjusted value. Since value depends upon the mean of the risk adjusted distribution, all you need to know is the real world mean and then subtract $\lambda$ * standard deviation. This is a much simpler approach than doing a full projection (and is not new, as it is effectively a restatement of the Sharpe ratio).

Consider the following example

| | | |
|---|---|---|
| $\mu$ | 6.618% | |
| $\mu^*$ | 3.827% | |
| $\sigma$ | 20% | |
| $\lambda$ | 14.0% | $\lambda = \frac{(\mu - \mu^*)}{\sigma}$ |

Consider a profit stream with

- expected profit: 0.264
- standard deviation: 0.180

(These are the parameters for the normal example in Section 6.1.)

The risk adjusted profit is 0.264 – $\lambda$ * 0.180
= 0.264 – 0.14* 0.180
= 0.264 – 0.025
= 0.239

The present value at the risk free rate is 0.239 / 1.06 = 0.225.

### 7.3 Cost of capital

Under simplified assumptions, stochastic discount factors can give the cost of capital.

If Z is normally distributed then risk based capital is a multiple of the standard deviation. Assuming we know the risk of ruin, we can replace standard deviation from the previous section with risk based capital. The $\lambda$ factor needs to be rescaled but this is straight forward. It just needs to be divided by the number of standard deviations implied by the risk of ruin.

If the risk of ruin is 0.5% then the risk based capital has 2.58 standard deviations. The cost of risk based capital is then $\lambda$ * 2.58 or 5.43%.

Using the numbers from the previous normally distributed example, the risk based capital is 0.463 for a one year projection (0.463= 2.58 * standard deviation = 2.58 * 0.180). For simplicity, assume the capital is only required at the end of the year.

The risk adjusted value is then

- The best estimate average profit
- Less the risk based capital times 5.43%
- Then discount at the risk free rate

In our example for a normally distributed risk:

| | |
|---|---|
| The best estimate average profit | 0.264 |
| Less the risk based capital times 5.43% | 0.483 * 5.43% =0.025 |
| Risk adjusted value | 0.264 – 0.025 = 0.239 |
| Discounted at risk free rate | 0.239 / 1.06 = 0.225 |

This gives a simple rule for the cost of capital and can be used in a deterministic projection.

This approach has been used for many years but it is worth knowing the assumptions that underpin it. It assumes:

- the cashflow Z is normally distributed. It does not work for skewed cashflows
- we know the number of standard deviations in the risk-based capital measure (that is, we know the risk of ruin)
- both the real world and risk neutral distributions are lognormal with known parameters (or, more precisely, there is constant relative risk aversion with a known coefficient)

If any of these assumptions don't hold then it is likely that a projection using the full range of stochastic discount factors would be required. In particular, the assumption that all risks are normal may not hold up in practice. Also, there is nothing to say that the risk aversion for financial risks is always constant.

The skewed example from Section 6.1 shows the limitations of just using capital as a measure of risk. The following table shows the cost of the risk compared to the standard deviation and the capital for the normal example and the skewed example.

*Table 7.2: Cost of risk as a percentage of standard deviation and capital*

| | Normal | Skewed | |
|---|---|---|---|
| Cost of risk | 0.025 | 0.021 | |
| Cost (excluding pres. value) | 0.027 | 0.022 | |
| | | | |
| Standard deviation | 0.180 | 0.144 | |
| Cost as % of standard dev | -15.0% | -15.3% | |
| | | | |
| Capital | 0.463 | 0.463 | With 0.5% risk of ruin |
| Cost as % of capital | -5.8% | -4.8% | |

Both risks have the same mean but the normal risk has a higher standard deviation. Standard deviation, however, is not always a good measure of risk as it doesn't pick up the addition skewness in the skewed risk. Proportionally, the cost of standard deviation is marginally in the skewed risk.

Both risks have the same mean and capital requirement using a 0.5% risk of ruin. However, the normal risk has a much higher cost of capital as it has a much heavier downside tail. This more than offset the additional upside under the normal risk.

The key point is that the cost of risk is not necessarily a straight proportion of the capital. The shape of the risk profile also has an impact. This feature is often picked up in the value of financial risks but is often overlooked in the value for other risks.

## 7.4 Log normal distributions and recovering the Black Scholes formula

If Z is log normally distributed then the shift in the distribution may also lead to another lognormal distribution.

Again, if $m_p = ve^{-\lambda[C_p+0.5\lambda]}$ then for a log normal distribution

| | Real world | Risk adjusted | Comment |
|---|---|---|---|
| Mean of ln(Z) | $\alpha$ | $\alpha - \beta\lambda$ | Shifted by $\beta\lambda$ |
| Standard deviation of ln(Z) | $\beta$ | $\beta$ | Unchanged |

A proof is given in the appendix. This is akin to the Black Scholes formula. The volatility assumption is maintained but the mean assumption is shifted downwards.

## 7.5 Company valuations and the cost of non-financial risk

Market consistent values are being used more often around the world for valuing financial services companies, but at present there is no agreed approach for including agency costs. Section 5.3 argued that one of the key drivers of agency costs is profit volatility from all risks. Stochastic discount factors could provide a generalised approach for quantifying the impact from total profit volatility.

This can be achieved in a number of ways:

1. Including a cost of risk based capital for non-financial risks. In Section 7.3 we saw that the cost of capital is about 5.4% (under certain simple assumptions). For non-financial

risks, we can deduct the cost of risk using a cost of the attributed risk based capital. As mentioned above, this can only be used if the risk is normally distributed and the stochastic discount factors take a neat form. The risk based capital is not necessarily the same as the regulatory capital and should allow for the actual magnitude of the underlying risks, including any internal diversification.

2. Alternatively, we could do a full stochastic projection and apply stochastic discount factors. This approach can be used for all risks, including asymmetric risks, and for any shaped stochastic discount factor. For example, operational risks are typically heavily skewed with a large likelihood of a small loss and a remote likelihood of a very large loss. Stochastic discount factors can derive a risk adjusted value for these risks.

In practice, a company valuation could be expressed as:

| | **$m** |
|---|---|
| Pure MCV value | A |
| Cost of non-financial risk | (B) |
| Net value | C = A - B |

To facilitate a comparison between companies, there are ways for expressing the cost of non-financial risk:

- x% of risk based capital, where x% is derived as in the example is section 7.3
- additional y% to the discount rate, where y% is routinely calibrated to give the desired total value
- a more sophisticated approach could say that cost is based on a particular risk aversion function

### 7.6 Enterprise risk management

Stochastic discount factors are useful for enterprise risk management. They can quantify all risks consistently and give a value to different risk management strategies.

The advantages for ERM from using stochastic discount factors are:

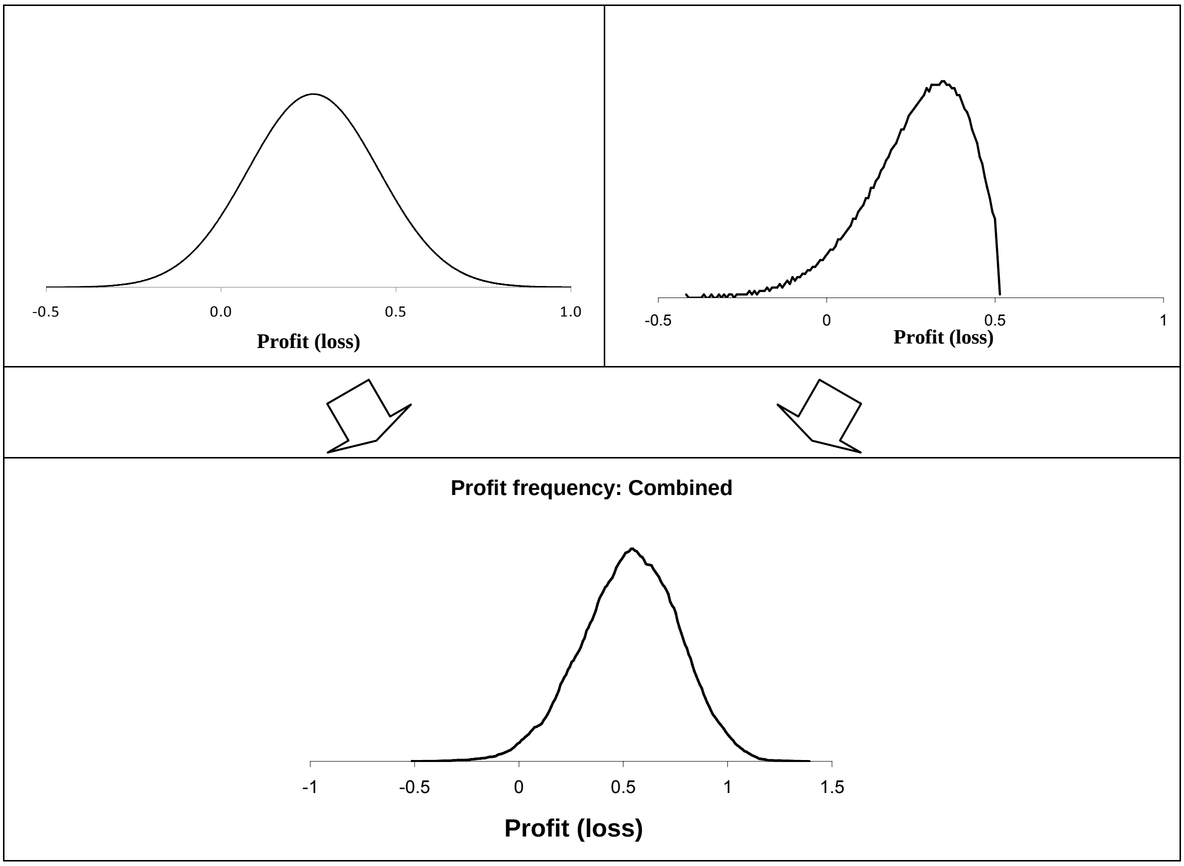
- all risks are valued
- financial and non-financial risks are valued consistently
- they can handle asymmetric profit profiles. ERM is often concerned with limiting the downside while maintaining the upside. Stochastic discount factors can handle this situation as they consider the full range of outcomes, but they give more weight to the downside.
- diversification benefits can be directly computed (see the next section)
- the unit cost of risk can be tracked over time by tracking the risk aversion implied in the shape of the stochastic discount rate curve

## 7.7 Diversification benefits

Stochastic discount factors can quantify a diversification impact.

Consider the two profit profiles from the start of Section 6. The following graphs show the profit profile if these two profiles are combined. This combined graph assumes the two risks are independent.

***Graph 7.1: Example of combining two risks***




The following table shows the value if these two profiles are combined. The value is greater than the sum of the parts because the stochastic discount factors are reordered to reflect the new profit profile. In the combined case, the highest stochastic discount factors are used when the combined profit is lowest, not when the profit from each individual risk is lowest.

***Table 7.3: Quantification of diversification benefits***

| | Normal | Skewed | Diversification | Combined | Comment |
|---|---|---|---|---|---|
| Best estimate profit | 0.264 | 0.264 | 0.000 | 0.528 | (a) Mean of real world distribution |
| PV at risk free rate | 0.249 | 0.249 | 0.000 | 0.498 | (b) = (a) / (1 + risk free rate) |
| Risk adjusted mean | 0.239 | 0.243 | 0.013 | 0.495 | (c) Mean of risk neutral dist. |
| **PV at risk free rate** | **0.225** | **0.230** | **0.012** | **0.467** | (d) = (c) / (1 + risk free rate) |
| Cost of risk | 0.025 | 0.021 | -0.013 | 0.033 | (e) = (a) - (c) |
| **PV of cost of risk** | **0.024** | **0.020** | **-0.013** | **0.031** | (f) = (b) - (d) |

This sort of combination may occur at a company level where the normal risk may be a product related risks such as mortality and the skewed risk may be a financial risk. This sort of diversification is one of the key drivers of value for financial services companies.

## 7.8 Risk adjusting historical returns

Section 5 showed that stochastic discount factors can convert a future outcome to a risk adjusted expected value.

Stochastic discount factors can also convert *past* observations into a risk adjusted measure. These factors give more weight to observations in low scenarios (for example when market returns were low, or if the company profit was low) and low factors in high scenarios.

For an example, see the paper by Farnsworth et al (2000), Performance Evaluation Using Stochastic Discount Factors.

## 7.9 Equivalent risk portfolios

(This application is less straight forward then the previous ones and can be skipped. It may be more of mathematical nicety than a genuine application.)

Using the methodology in this paper, the value of any profit profile is the value of an "equivalent risk portfolio".

The equivalent risk portfolio is a combination of the risky market asset and a risk free asset, and has the same range of outcomes as the given risk.

For a financial risk, the equivalent risk portfolio is the same as the replicating portfolio. It provides the same outcomes in the same market conditions. This is the principle underlying a market consistent valuation.

For a non-financial risk, the equivalent risk portfolio provides the same range of outcomes with the same likelihood. Importantly, however, the link with the market conditions is broken. The non-financial risk and the equivalent risk portfolio provide the same outcomes, but not necessarily at the same time.

In all cases, the cost of risk is directly related to the level of risky assets in the equivalent portfolio.

This is not straight forward and a full explanation is beyond the scope of this paper. (It basically works by recognising that a risk neutral distribution can be derived from options, and options are a mixture of risky and risk free assets.)

As a high level example, the following table shows the composition of a portfolio that has the same risk profile as the normal and skewed example from Section 6, and the combined profile from Section 7.7.

***Table 7.4: Equivalent risk portfolios***

| | **Normally distributed profile** | **Skewed profile** | **Combined profile** |
|---|---|---|---|
| **Equivalent risk portfolio** | | | |
| Risky market asset | 0.848 | 0.695 | 1.096 |
| Risk free asset | (0.623) | (0.465) | (0.629) |
| **Total value** | 0.225 | 0.230 | 0.467 |

This gives a way of explaining the riskiness of a particular profit profile. For example, senior management may not readily understand the risk inherent in the skewed example. However, they may understand "This has the same risk as taking $230 of your own money, borrowing $465 and buying $695 of shares."

Knowing this breakdown does not help with managing the risk. Taking the skewed example, we can't then sell $695 of shares to hedge the risk.

The cost of risk is directly related to the equivalent market exposure. Mathematically, the cost of risk is present value of the market risk premium times the equivalent exposure to the market asset. This is shown in Table 7.5.

***Table 7.5: Deriving the cost of risk from the exposure of equivalent risk portfolio***

| | **Normally distributed profile** | **Skewed profile** | **Combined profile** |
|---|---|---|---|
| Market asset exposure of equivalent risk portfolio | 0.848 | 0.695 | 1.096 |
| Market risk premium[1] | 3.00% | 3.00% | 3.00% |
| Market exposure times the risk premium | 0.025 | 0.021 | 0.033 |
| Present value (The result is the same as cost of risk) | 0.024 | 0.020 | 0.031 |

Note 1: The market equity premium is the expected return on market asset (9% in this example) less the risk free rate (6%).

As mentioned earlier, this may be more of a mathematical nicety than a genuine application. What it does show, however, is that value of any risk profile is the value of an equivalent risk portfolio. The cost of risk is then the equivalent market exposure times the equity premium.

# 8 Limitations

There are a number of limitations with the approach described in this paper. Some are described below.

## 8.1 A real world distribution

The approach assumes that a real world distribution can be derived for company profit (or for what ever risk is being valued). While this is an ideal goal, it is not always straight forward in practice.

## 8.2 Agency costs

There is nothing to say that the market risk aversion explains all of the difference between a pure MCV value and the market capitalisation of a company. There may well be other costs that are not included in the models.

The approach in this paper will help if the main driver of agency costs is profit volatility.

## 8.3 Not arbitrage free

The approach fails to meet one of the key requirements for a market consistent value as it is not arbitrage free for non-financial risks.

The non-financial risks, by definition, cannot be hedged using market assets. Therefore, knowing that the cost of non-financial risk doesn't tell you how much you need to pay to avoid that risk. To remove the risk could cost more or less than that and some costs may not be removable at all.

In comparison, the cost of a financial risk has real meaning: it is the market cost for hedging that risk.

## 8.4 Deriving the market's view of the real world distribution

The approach described in this paper assumes that you can find the market's view for the real world distribution for the future level of the market. This will not be a trivial exercise. Complex time series models can be developed but there is no guarantee that these will reflect the market's views.

## 8.5 Market and company risk aversion

The approach assumes the company's risk aversion is the same as the market risk. There is, of course, no guarantee that this will hold. At best, it serves as a benchmark.

This is not necessarily a bad thing. It is a way that a company can align its risk preferences with the market.

## 8.6 Recovering the value of financial risks

In practice, there will be more than one asset class and more than one market. There is no guarantee that applying a single set of stochastic discount factors to the real world distribution for all financial assets will then produce the same value as using the risk neutral assumptions, particularly as the number of asset classes increases.

In this case, it is better to value financial risks using the risk neutral assumptions and then overlay the other risks, combined with stochastic discount factors, at the company level.

## 8.7 Extension to multi-periods

The approach described in this paper works for single periods. For multi periods it is necessary to combine the individual periods together to reflect the correlation between years appropriately. This is because, for example, the correlation for equity markets between two successive years is different to the correlation between years for other risks.

To get around this limitation, the company profit should become the accumulated company profit at the end of the projection period. The accumulation should be at the risk free rate. The stochastic discount factors should then relate to the implied risk aversion for a market asset at the end of the same projection period.

In practice, it may be difficult to derive a long term view for either Q or P. For example, there is unlikely to be sufficient information for long term options that are deep in or out of the money.

## 8.8 Not good for extreme risks

The approach may not work for extreme risks as it is unlikely we can derive a reliable measure of risk aversion at the edges of the market. This could be a major limitation because many operational risks have low likelihoods but potentially high costs.

## 9 Conclusions

The goal of this paper was to propose a general formula for valuing any set of uncertain future profits on a consistent basis.

To meet this goal, there is a general formula for a future uncertain outcome Z with a real world distribution P(z), value = Σm.z.P(z) (or E(mZ)), where m is the stochastic discount factors derived from the value of financial assets.

This formula meets the criteria outlined at the start of this paper because:

- it produces a risk adjusted value; the higher the risk the lower the value (all other things being equal). The formula gives more weight to the downside than to the upside, meaning the greater the spread of outcomes, especially on the downside, the lower the value
- it applies for all shapes of profit distributions
- it is useful for enterprise risk management purposes as it looks at the impact on value from all sources of risk.
- it values financial risks consistently with observable market values for similar risks, as it uses the market implied risk aversion as a key input
- it can be explained in general terms, either as a cost of capital for simple risks or, for more complex cases, as a risk aversion function

This paper came from a desire to include non-financial risk in a market consistent valuation. What emerged from it was a better understanding of what the cost of financial risk actually represents, and an idea for how this cost can have a wide range of applications.

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The literature is the wide range of names for similar concepts, as shown in the table below.

| **Real world distribution** | **Risk neutral distribution** | **Stochastic discount factors** |
|---|---|---|
| is similar to:<br>Realistic distribution<br>Subjective probability<br>Implied probability<br>Accurate probability<br>Beliefs<br>P - probability | is similar to:<br>Objective probability<br>Distorted probability<br>Pricing kernel<br>State price density<br>Q – probability | is similar to:<br>Risk distortion<br>Risk transformation<br>Pricing kernel<br>Preferences<br>Risk appetite<br>Risk aversion<br>Deflators<br>State price deflators<br>Marginal utility<br>Marginal inter-temporal rate of substitution |

## Appendix 1: Deriving risk neutral distributions

Market consistent values work by finding some factor "Q" that we can apply to different future profits, and then discount at the risk free rate. The risk adjusted value is then the sum of all these across all possible cashflows.

That is, they use some function Q(x) such that $V = \int Z_x vQ(x)dx$.

For this to work, vQ(x)dx needs to be the value of an asset that pays 1 when the market is between x and x+dx and zero at all other times.

A key advantage of this function Q(x) for valuing financial risks is that it can be readily derived using option prices.

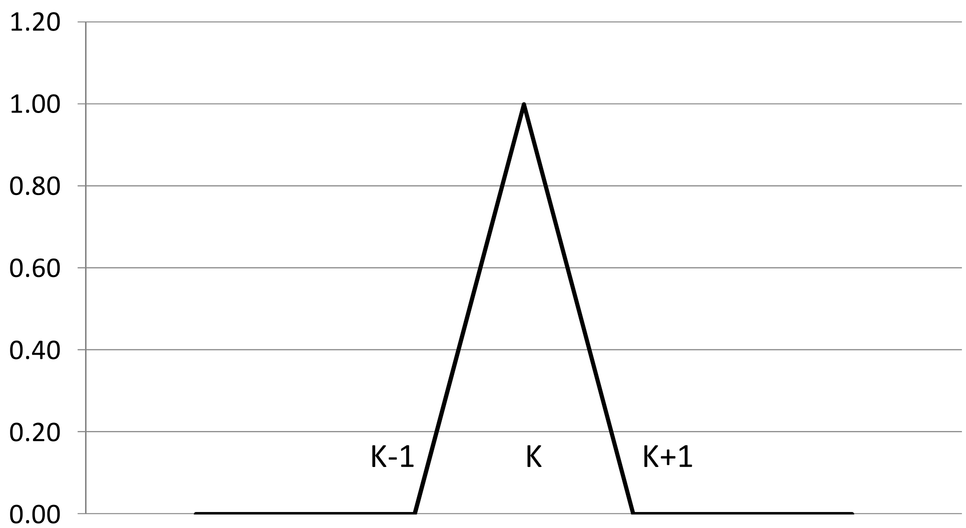
In 1978, Breeden and Litzenberger demonstrated that vQ is the second derivative of the option price to the strike price.

To show this, consider the following portfolio using three options.

- Buy 1 call option with a strike price of K-1
- Sell 2 call options with a strike price of K
- Buy 1 call option with a strike price of K+ 1

Graph A1.1 shows the pay off at different future levels of the market.

***Graph A1.1: Payout of options with K-1, K and K+1 strike prices***




This portfolio pays 1 when the market value is K and zero at all other times, except for a triangle between K-1 and K+1. This triangle can be minimised by using strike prices K-δx and K +δx and letting δx go to zero. In this case, we need to buy 1/δx times the number of options.

As δx tends to zero, the value of this portfolio must be vQ(x)dx, as it pays 1 when the market is at level K and zero at all other times.

**Using the market implied risk aversion to value all risk**

Mathematically,

Let C(K) be a call option with a strike price of K

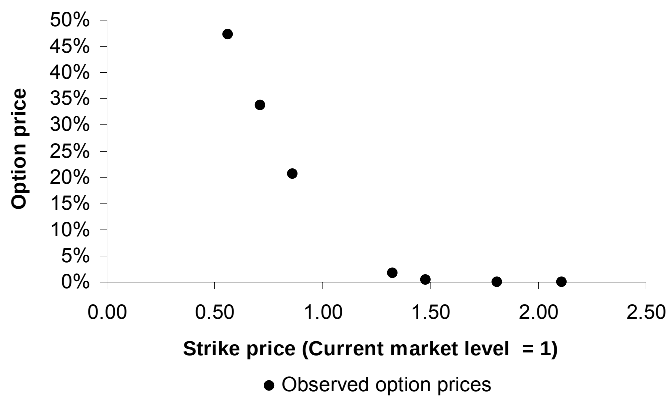
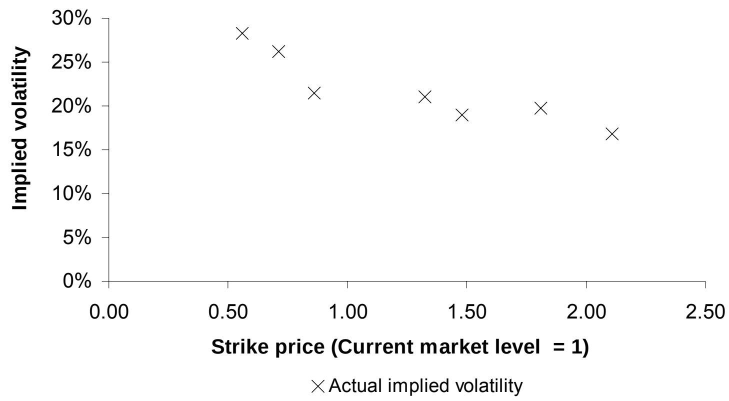
vQ(K)dx = {C(K-dx ) – 2C(K) + C(K+dx) } /dx
= {C(K-dx ) – C(K)} /dx - {C(K) – C(K+dx )} / dx
= -(first order change at K - first order change at K-1)

So vQ(K) = -(first order change at K - first order change at K-1) / dx
= second order derivative of option price with respect to K

This relationship doesn't rely on any particular pricing formula for call options. It works for Black Scholes or any other option pricing formula.

To get a second derivative, the option price needs to be a smooth function of the strike price. This is unlikely to be the case in practice, and so some sort of formula will be required to keep things smooth.

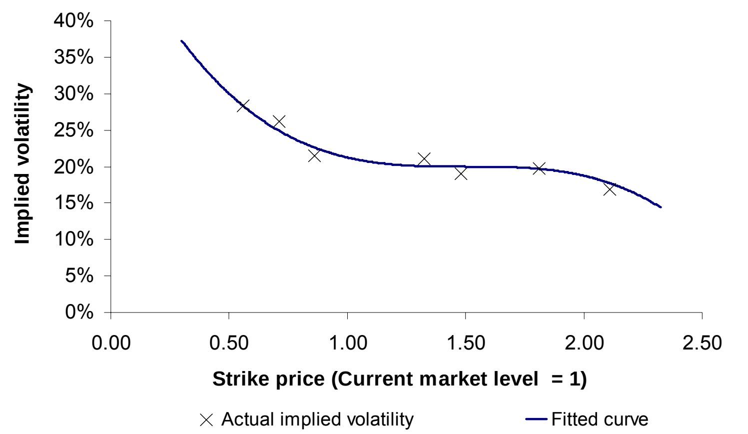
One approach is shown in the following set of graphs. The numbers in these examples are for illustration only:

| | |
|---|---|
| **Step 1**<br>Get option price for different strike prices |  |
| **Step 2**<br>Get the implied volatility for each option price. This requires an option formula such as Black Scholes |  |

**Step 3**

Fit a smooth curve to the volatility.

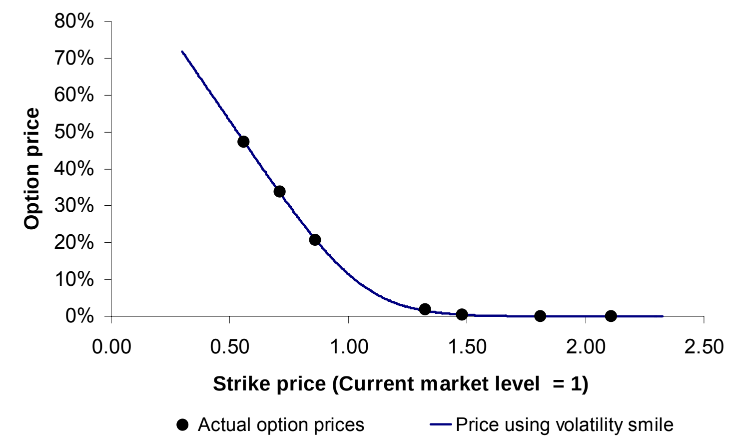
Often, volatility increases with lower strike prices and decreases with higher strike prices, hence the curve is typically known as a “volatility smirk”. (A volatility smile has increasing volatility for both lower and higher strike prices)




**Step 4**

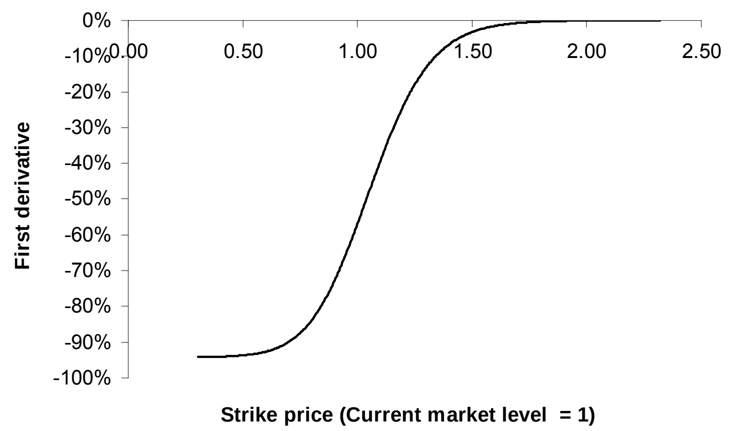
Get option prices using implied volatilities curve. This requires putting the smoothed volatilities back into the option pricing formula

Getting a smooth volatility smirk first and then getting a smooth option price is often easier than fitting a smooth curve to the option prices in the first instance




**Step 5**

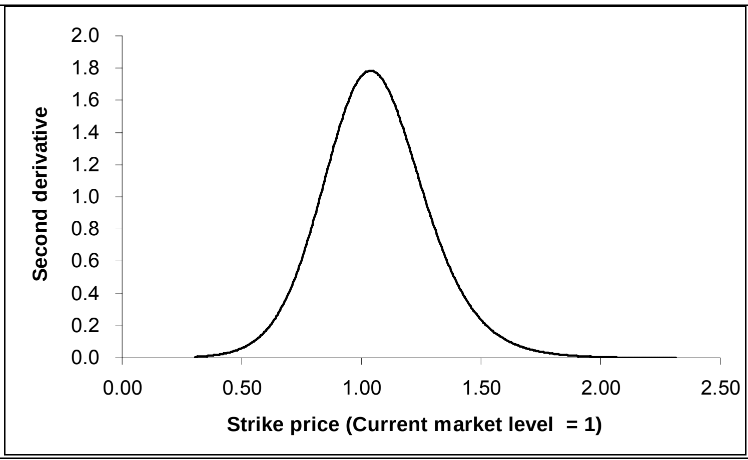
Get first derivative

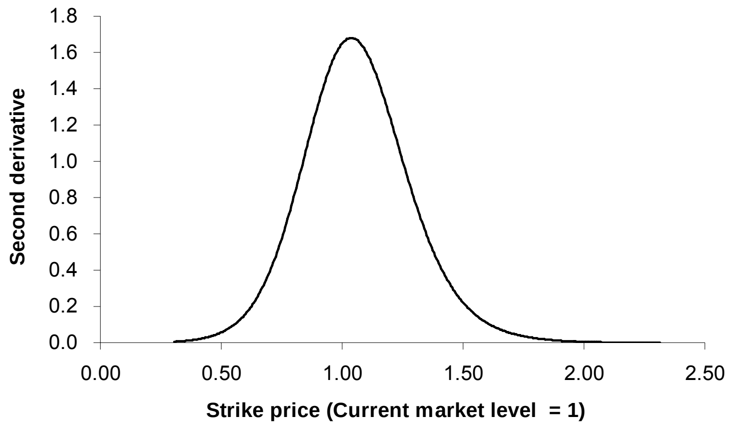



**Step 6**

Get second derivative

= vQ(x)



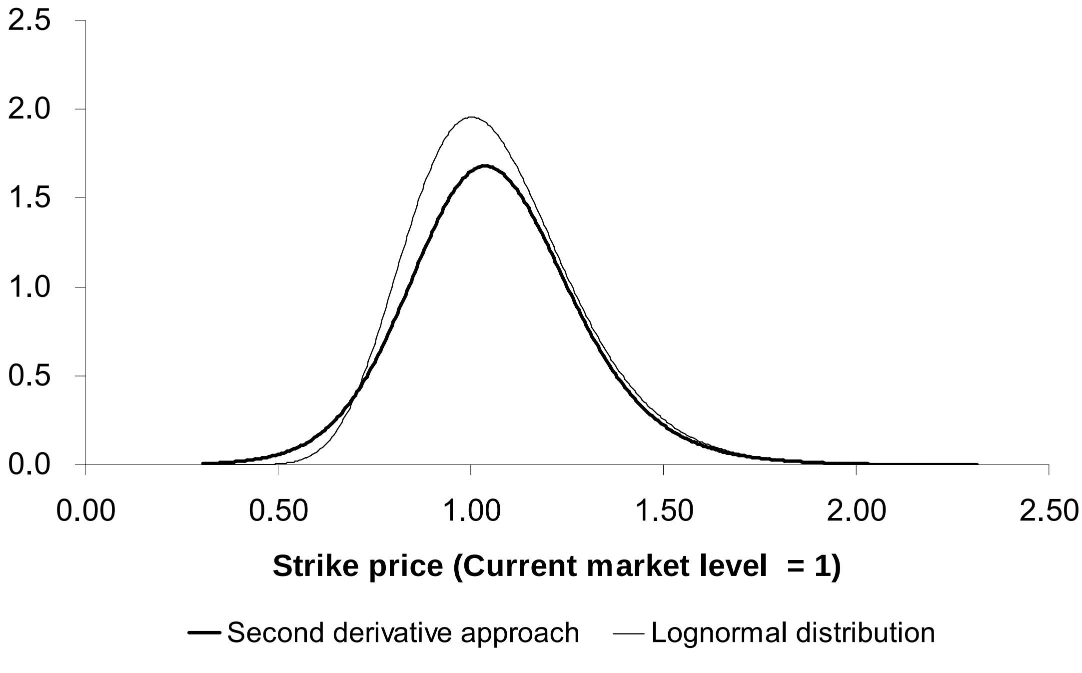
| | |
|---|---|
| **Step 7**<br><br>Divide by v to the Q(x) |  |

This is just one way of deriving Q. Other ways include fitting parameters to a functional form for either the option price formula or for the actual Q(x) function.

For detailed examples of this concept see Jackwerth (1997), Aït-Sahala and Lo (2000) and Chang and Tabak (2002).

While Black Scholes assumes a log normal distribution for share prices, using the Breeden and Litzenberger approach does not necessarily lead to a log normal distribution for the risk neutral assumption. If the volatility curve was flat using the Black Scholes formula, then Q(x) would be the same as a lognormal distribution. If volatility increases for lower strike prices then the risk neutral distribution has fatter left hand tail than under a lognormal distribution. The following graph compares the above graph, which has the volatility smirk, with a lognormal curve with a constant volatility.

***Graph A1-1: Curve with volatility smirk compared to curve with constant volatility.***



## Appendix 2: More on risk aversion

A useful measure of utility is to look at the curvature of the marginal utility, rather than the absolute level.

The most well known measures of this curvature of marginal utility were introduced by John Pratt (1964) and Kenneth Arrow (1965): absolute risk aversion and relative risk aversion.

Absolute risk aversion (ARA): $-\frac{U''(x)}{U'(x)}$

Relative risk aversion (RRA): $-\frac{U''(x)}{U'(x)}x$

Arrow postulated that risk aversion should not necessarily be constant. He suggested that individuals should demonstrate decreasing absolute risk aversion and increasing relative risk aversion.

In simple terms, if all people have the same *absolute* risk aversion then all people will pay the same dollar amount to avoid the same dollar amount of risk. If absolute risk aversion decreases as wealth increases, then a richer person would pay less to avoid the same dollar amount of risk. This may make sense, as a wealthy person may be less sensitive to losing \$1,000 than a poor person would be.

In simple terms, if all people have the same *relative* risk aversion then all people will pay the same proportion of wealth to avoid the same relative amount of risk. If relative risk aversion increases as wealth increases, then a richer person would pay more to avoid the same relative amount of risk. This may make sense, as a wealthy person may be more sensitive to losing 10% of their wealth than a poor person would be.

The following table shows some common functions for utility and gives the absolute risk aversion and relative risk aversion. The first three forms are quite common and have effectively been designed to give a meaningful risk aversion parameter.

***Table A2.1: Common utility functions and resulting risk aversion***

| **Function U(x)** | **Absolute risk aversions** | **Relative risk aversion** | **Comments** |
|---|---|---|---|
| $-\frac{1}{\gamma}e^{-\gamma x}$ | $\gamma$ | $\gamma x$ | Negative exponential utility function<br>Constant absolute risk aversion |
| $\frac{1}{1-\gamma}x^{1-\gamma}$ | $\frac{\gamma}{x}$ | $\gamma$ | Power utility function<br>Constant relative risk aversion<br>**Implied by Black Scholes option pricing model** |
| ln(x) | $\frac{1}{x}$ | 1 | Special case of power utility but with $\gamma = 1$ |
| $\frac{1}{1-\gamma}(x+\eta)^{1-\gamma}$ | $\frac{\gamma}{x+\eta}$ | $\frac{\gamma}{1+\frac{\eta}{x}}$ | Relative risk aversion increases with increase in wealth |
| $Ax+Bx^2$ | $\frac{-2B}{A+2Bx}$ | $\frac{-2Bx}{A+2Bx}$ | Quadratic function<br>B needs to be negative (but not too negative) to give a positive risk aversion |

## Deriving the market's implied risk aversion

It is possible to derive the market's risk aversion by recognising that the stochastic discount factor is proportional to the marginal utility (see Jarvis et al for a proof).

Since the stochastic discount factor equals $\frac{Q}{P}v$ and is proportional to the marginal utility, then (because v is a constant factor it drops out and after some simplifying maths)

Absolute risk aversion (ARA) = $-(\frac{Q'}{Q} - \frac{P'}{P})$

Relative risk aversion (RRA) = $-(\frac{Q'}{Q} - \frac{P'}{P}).x$

Therefore, if we can model market view for Q and P we can derive the market implied risk aversion.

There is a large amount of literature on deriving risk aversion from option prices and models for future market returns. See:

- Aït-Sahalia and Lo, 2000, Non Parametric Risk Managements and Implied Risk Aversion
- Jackwerth, 1997, Recovering Risk Aversion from Option Prices and Realised Returns
- Ziegler, 2003, Why Does Risk Aversion Smile?

Common features of these investigations are:

- risk aversion is not flat and appears to "smile". That is, it increases for both high and low returns
- the smile may be crooked at some points (that is, there may be a small bump around the middle)
- risk aversion may even turn negative at the bottom of the smile

## A three-way relationship

The following is an important relationship.

If we know two of

i. Risk aversion
ii. Risk real world distribution (P)
iii. Risk neutral distribution (Q)

Then we can derive the third.

Of these three, only one component is readily observable: the risk neutral distribution, as it can be derived form option prices.

Q is not necessarily stable. The question becomes, do these movements come from movements in risk aversion or from movements in P.

### Using the market implied risk aversion to value all risk

Economists and policy makers such as central bankers tend to view risk aversion as more stable. If Q moves it is because P is moving. They track the changes in option prices to give an idea of future market movements.

Those with a statistical background may view P as stable and movements in Q arise from changes in risk aversions. For example, the 2007 / 2008 sub prime crisis is the US increased risk aversion and decreased assets values in other markets, even through those other markets did not necessarily down grade their profit outlooks.

In practice, it is likely to be a combination of both effects.

## Appendix 3: Risk neutral probability in Black Scholes

Back Scholes assumes that the underlying asset price follows a lognormal probability distribution.

The standard Black Scholes formula for a European call option on a stock that pays no dividends is

$C = X.N(d_1) – Ke^{-rt}N(d_2)$

| | |
|---|---|
| $d1 = \dfrac{\ln(\frac{X_t}{K}) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$<br>$d2 = d_1 - \sigma\sqrt{t}$ | Where<br><br>X = the current share price<br>$X_t$ = the share price after t years<br>K = the strike price<br>r = the risk free rate<br>σ = the volatility of the share price returns<br>t = the period to expiry |

If $X_t$ is the share price at the end of time t and is lognormally distributed with parameters μ and σ (that is, $\ln(X_t)$ is normally distributed with mean μ and standard deviation σ).

Under any risk neutral distribution, the present value of the expected value of a call option is $PV(Max(X_t – K, 0))$

Hull (1997), Chapter 12A shows that if $X_t$ is log normal then the present value of the expected value is $PV\{E(X_t))N(d_1) – K\,N(d_2)\}$

Where

$$d_1 = \frac{\ln(\frac{E(X_t)}{K}) + \frac{\sigma^2 t}{2})}{\sigma\sqrt{t}}$$

$$d_2 = \frac{\ln(\frac{E(X_t)}{K}) - \frac{\sigma^2 t}{2})}{\sigma\sqrt{t}}$$

Substituting in for $E(X_t) = X e^{\mu t + \frac{\sigma^2 t}{2}}$

$$d_1 = \frac{\ln(\frac{Xe^{\mu t + 0.5\sigma^2 t}}{K}) + \frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}$$

$$= \frac{\ln(\frac{X}{K}) + \mu t + \frac{\sigma^2 t}{2} + \frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}$$

$$d_2 = \frac{\ln(\frac{X}{K}) + \mu t + \frac{\sigma^2 t}{2} - \frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}$$

Now, PV(E($X_t$)) = $e^{-rt} X e^{\mu t + 0.5\sigma^2 t}$

We know that to be consistent with market value, PV(E($X_t$) = X (the present vale of anything is the current value). To get this, So r = $\mu$ + 0.5$\sigma^2$

Substituting in for r

$$d_1 = \frac{\ln(\frac{X}{K}) + rt + \frac{\sigma^2 t}{2}}{\sigma\sqrt{t}}$$

$$d_2 = \frac{\ln(\frac{X}{K}) + rt - \frac{\sigma^2 t}{2}}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}$$

From the payout for an option:

Present value of the expected value is

= PV(Max($X_t$ – K, 0))

= PV{E($X_t$))N($d_1$) – K N($d_2$)}

= XN($d_1$) – K $e^{-rt}$ N($d_2$)

This is the same as the Black Scholes formula.

## Appendix 4: Proof of functional form of stochastic discount factor under log normal assumptions for risk neutral and real world distributions

Lognormal pdf = $\frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$

Real world pdf = $\frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$ ……. (1)

Risk neutral pdf = $\frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu^*)^2}{2\sigma^2}}$ ……. (2)

$$\frac{\text{Risk neutral probability}}{\text{Real world probality}} = \frac{\frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu^*)^2}{2\sigma^2}}}{\frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}}$$ …….(3) = (2)/(1)

$$\frac{\text{Q}}{\text{P}} = \frac{e^{-\frac{(\ln(x)-\mu^*)^2}{2\sigma^2}}}{e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}}$$ …… cancel out first term

$$= e^{-\frac{(\ln(x)-\mu^*)^2-(\ln(x)-\mu)^2}{2\sigma^2}}$$ ……because $y^m/y^n = y^{(m-n)}$

$$= e^{-\frac{(\ln(x)^2-2\ln(x)\mu^*+(\mu^*)^2-(\ln(x)^2-2\ln(x)\mu+\mu^2)}{2\sigma^2}}$$ …… expand out square terms

$$= e^{-\frac{(\ln(x)^2-2\ln(x)\mu^*+(\mu^*)^2-\ln(x)^2+2\ln(x)\mu-\mu^2)}{2\sigma^2}}$$ …. Multiple second brackets by -1

$$= e^{-\frac{(-2\ln(x)(\mu^*-\mu)\ +(\mu^*)^2-\mu^2)}{2\sigma^2}}$$ .. $\ln(x)^2$ cancel out and bring together 2ln(x) terms

$$= e^{-\frac{(-2\ln(x)(\mu^*-\mu)+(\mu^*-\mu)(\mu^*+\mu))}{2\sigma^2}}$$ ….. $\mu^{*2} - \mu^2 = (\mu^* - \mu).(\mu^* + \mu)$

$$= e^{-\frac{(\mu^*-\mu)(-2\ln(x)+(\mu^*+\mu))}{2\sigma^2}}$$ .. pull out ($\mu^*$ - $\mu$)

$$= e^{-\frac{(\mu-\mu^*)(2\ln(x)-(\mu+\mu^*))}{2\sigma^2}}$$ …… multiple both brackets by -1, so ($\mu^*$ - $\mu$) becomes ($\mu$ - $\mu^*$)

$$= e^{-\left[\frac{(\mu-\mu^*)}{\sigma}\right]\left[\frac{(2\ln(x)-(\mu+\mu^*))}{2\sigma}\right]}$$ …. Split out $\sigma$

$= e^{-\left[\frac{(\mu-\mu^*)}{\sigma}\right]\left[\frac{(\ln(x)-0.5^*(\mu+\mu^*))}{\sigma}\right]}$ ….. (4) divide second bracket by 2

Now

$\lambda = \dfrac{(\mu - \mu^*)}{\sigma}$ …… (5)

So $\mu - \mu^* = \lambda\sigma$ ….. rearrange

$\mu^* = \mu - \lambda\sigma$ ….. (6) rearrang

So
$\mu + \mu^*$
$= \mu + \mu - \lambda\sigma$ Insert (7) for $\mu^*$
$= 2\mu - \lambda\sigma$ ……..(8)

So

$= e^{-\left[\frac{(\mu-\mu^*)}{\sigma}\right]\left[\frac{(\ln(x)-0.5^*(\mu+\mu^*))}{\sigma}\right]}$ ……. From equation (4)

$= e^{-[\lambda]\left[\frac{(\ln(x)-0.5^*(2\mu-\lambda\sigma))}{\sigma}\right]}$ Substitute the first term for λ and insert equation (8) for $\mu + \mu^*$

$= e^{-[\lambda]\left[\frac{(\ln(x)-\mu+0.5^*\lambda\sigma))}{\sigma}\right]}$ ……. Expand the brackets.

$= e^{-[\lambda]\left[\frac{(\ln(x)-\mu)}{\sigma}+\frac{0.5\lambda\sigma}{\sigma}\right]}$ . split out term

$= e^{-[\lambda]\left[\frac{(\ln(x)-\mu)}{\sigma}+0.5\lambda\right]}$ .

So

$\dfrac{\mathrm{Q}}{\mathrm{P}} = e^{-[\lambda]\left[\frac{(\ln(x)-\mu)}{\sigma}+0.5\lambda\right]}$

So

$\dfrac{\mathrm{Q}}{\mathrm{P}}\mathrm{v} = \mathrm{v}e^{-[\lambda]\left[\frac{(\ln(x)-\mu)}{\sigma}+0.5\lambda\right]}$

## Appendix 5: Proof of shift in normal and lognormal distributing under constant relative risk aversion

Under constant relative risk aversion, the stochastic discount factor for a percentile p is $m(p) = ve^{-[\lambda][C_p+0.5\lambda]}$

If Z is normally distributed with mean α and standard deviation β, then the function for m.Z is

$$e^{-[\lambda][Cp+0.5\lambda]} \cdot \frac{1}{\beta\sqrt{2\pi}} e^{-\frac{(z-\alpha)^2}{2\beta^2}}$$

$$\frac{1}{\beta\sqrt{2\pi}} e^{-(\frac{(z-\alpha)^2}{2\beta^2}+[\lambda][C_p+0.5\lambda])}$$

$$\frac{1}{\beta\sqrt{2\pi}} e^{-(\frac{(z-\alpha)^2}{2\beta^2}+[\lambda]\left[\frac{(z-\alpha)}{\beta}+0.5\lambda\right])}$$ as $C_p = \frac{z-\alpha}{\beta}$ corresponds to the standard normal variable

$$\frac{1}{\beta\sqrt{2\pi}} e^{-(\frac{(z-\alpha)^2}{2\beta^2}+[\lambda\beta]\left[\frac{2(z-\alpha)+\lambda\beta^2}{2\beta}\right])}$$

$$\frac{1}{\beta\sqrt{2\pi}} e^{-(\frac{(z-\alpha)^2+2\lambda\beta z-2\lambda\beta\alpha+(\lambda\beta)^2}{2\beta^2})}$$

$$\frac{1}{\beta\sqrt{2\pi}} e^{-(\frac{(z^2-2z\alpha+\alpha^2+2\lambda\beta z-2\lambda\beta\alpha+(\lambda\beta)^2}{2\beta^2})}$$

$$\frac{1}{\beta\sqrt{2\pi}} e^{-(\frac{(z^2-2z(\alpha-\lambda\beta)+\alpha^2-2\lambda\beta\alpha+(\lambda\beta)^2}{2\beta^2})}$$

$$\frac{1}{\beta\sqrt{2\pi}} e^{-(\frac{(z^2-2z(\alpha-\lambda\beta)+(\alpha-\lambda\beta)^2}{2\beta^2})}$$

$$\frac{1}{\beta\sqrt{2\pi}} e^{-(\frac{(z-(\alpha-\lambda\beta))^2}{2\beta^2})}$$

= the pdf for a normal variable with mean α - λβ and standard deviation β

The proof for a log normal function is similar but $\frac{\ln(z) - \alpha}{\beta}$ corresponds to the standard normal variable with E(ln(Z)) = α and standard deviation(ln(Z)) = β

The function for m.Z is

$$e^{-[\lambda][C_p+0.5\lambda]} \cdot \frac{1}{\beta z\sqrt{2\pi}} e^{-\frac{(\ln(z)-\alpha)^2}{2\beta^2}}$$

$$\frac{1}{\beta z\sqrt{2\pi}} e^{-(\frac{(\ln(z)-\alpha)^2}{2\beta^2}+[\lambda][C_p+0.5\lambda])}$$

$$\frac{1}{\beta z\sqrt{2\pi}} e^{-(\frac{(\ln(z)-\alpha)^2}{2\beta^2}+[\lambda]\left[\frac{(\ln(z)-\alpha)}{\beta}+0.5\lambda\right])}$$ as $C_p = \frac{\ln(z) - \alpha}{\beta}$ is the standard normal variable

$$\frac{1}{\beta z\sqrt{2\pi}} e^{-(\frac{(\ln(z)-\alpha)^2}{2\beta^2}+[\lambda\beta]\left[\frac{2(\ln(z)-\alpha)+\lambda\beta^2}{2\beta}\right])}$$

$$\frac{1}{\beta z\sqrt{2\pi}} e^{-(\frac{(\ln(z)-\alpha)^2+2\lambda\beta\ln(z)-2\lambda\beta\alpha+(\lambda\beta)^2}{2\beta^2})}$$

$$\frac{1}{\beta z\sqrt{2\pi}} e^{-(\frac{(\ln(z)^2-2\ln(z)\alpha+\alpha^2+2\lambda\beta\ln(z)-2\lambda\beta\alpha+(\lambda\beta)^2}{2\beta^2})}$$

$$\frac{1}{\beta z\sqrt{2\pi}} e^{-(\frac{(\ln(z)^2-2\ln(z)(\alpha-\lambda\beta)+\alpha^2-2\lambda\beta\alpha+(\lambda\beta)^2}{2\beta^2})}$$

$$\frac{1}{\beta z\sqrt{2\pi}} e^{-(\frac{(\ln(z)^2-2\ln(z)(\alpha-\lambda\beta)+(\alpha-\lambda\beta)^2}{2\beta^2})}$$

$$\frac{1}{\beta z\sqrt{2\pi}} e^{-(\frac{(\ln(z)-(\alpha-\lambda\beta))^2}{2\beta^2})}$$

= the pdf for a log normal variable with parameters α - λβ and β