Living Life Optimally with a
Mean-Reverting Price of Risk
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Objective

• The objective of this paper was to analyse the behaviour of an investor when:
  – the risk premium is mean-reverting and is correlated with the risky asset.
  – In addition, event risks (jumps) exist in the risky asset.
Motivation

• Poterba and Summers (1988) have observed that serial negative correlation seems to exist for returns in the longer horizon in the US and 17 other countries, suggesting mean-reversion.

• It is well known that stock prices are susceptible to sudden changes.
Kim and Omberg

- Risky asset

\[
\frac{dS}{S} = \alpha_t dt + \sigma dZ_1
\]

- Risk premium

\[
X_t = \frac{\alpha_t - r}{\sigma} \quad \text{and} \quad X_t = -\kappa_x (X_t - \bar{X}) dt + \sigma_x dZ_2
\]
Kim and Omberg

• The investor’s utility

\[ U(W) = \frac{W^{1-\gamma}}{1-\gamma} \]

• Objective of the investor is to maximise expected terminal-utility.
Analytic Solution

- Asset allocation can be broken up into two parts: myopic demand (Merton ratio) and intertemporal demand.

\[
\theta(X_t, \tau) = \frac{X_t}{\sigma} + \frac{\rho \sigma_x [C(\tau) X_t + B(\tau)]}{\sigma} 
\]

- Myopic demand
- Intertemporal demand
Myopic and Non-myopic behaviour

• An investor is said to be myopic if he/she only considers this current period when making investment decisions.

• Non-myopic behaviour, on the other hand, occurs when the investor considers the problem as a whole. Intertemporal demand is part of non-myopic behaviour.
Analytic Solution

- Asset allocation can be broken up into two parts: myopic demand (Merton ratio) and intertemporal demand.

\[
\theta(X_t, \tau) = \frac{X_t}{\sigma \gamma} + \frac{\rho \sigma_x [C(\tau)X_t + B(\tau)]}{\sigma \gamma}
\]

- Myopic demand
- Intertemporal demand
Example

• Suppose the correlation is negative and the investor has a risk aversion of 4.
Example

<table>
<thead>
<tr>
<th>t = 0</th>
<th>t = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Return (Drift)</strong></td>
<td></td>
</tr>
<tr>
<td>7%</td>
<td>10%</td>
</tr>
<tr>
<td></td>
<td>5%</td>
</tr>
<tr>
<td><strong>Share price</strong></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>104</td>
</tr>
<tr>
<td></td>
<td>108</td>
</tr>
</tbody>
</table>

Risk-free = 6%

<table>
<thead>
<tr>
<th></th>
<th>Myopic investor: 25%</th>
<th>Non-Myopic investor: 30%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>105.6</td>
<td>105.5</td>
</tr>
<tr>
<td></td>
<td>106.5</td>
<td>106.6</td>
</tr>
</tbody>
</table>

Slide 10
### Example

<table>
<thead>
<tr>
<th></th>
<th>Myopic</th>
<th>Non-myopic</th>
</tr>
</thead>
<tbody>
<tr>
<td>t = 1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>104</td>
<td>114.42</td>
<td>114.31</td>
</tr>
<tr>
<td>33%</td>
<td>112.75</td>
<td>112.65</td>
</tr>
<tr>
<td>t = 2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>118</td>
<td>113.19</td>
<td>113.30</td>
</tr>
<tr>
<td>113</td>
<td>112.40</td>
<td>112.51</td>
</tr>
<tr>
<td>116</td>
<td>112</td>
<td></td>
</tr>
<tr>
<td>112</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Higher utility
Wu (2003)

• Wu considers the case when jump occurs.

\[ \frac{dS}{S} = (\alpha_t - \lambda g) dt + \sigma dZ_t + (e^q - 1) dQ \]
Jump Demand

- The asset allocation is:

\[ \theta(X_t, \tau) = \frac{X_t}{\sigma\gamma} + \frac{\rho \sigma_x [C(\tau)X_t + B(\tau)]}{\sigma\gamma} + \frac{\lambda \hat{g}}{\sigma\gamma} \]

- Negative jumps reduce the asset allocation.
Wachter (2002)

• Considers consumption without jumps.

• Consumption is important because one of the main reasons we invest is to consume the wealth.
Weighted Average Formula

- Wachter shows that the asset allocation is actually the weighted average of future consumption value.
Impact of consumption

• It follows that consumption actually impacts the asset allocation.
Model

• Risky asset

\[
\frac{dS}{S} = (\alpha_t - \lambda g) dt + \sigma dZ_1 + (e^q - 1) dQ
\]

• Risk premium

\[
X_t = \frac{\alpha_t - \lambda g - r}{\sigma} \quad \text{or} \quad X_t = -\kappa_x (X_t - \bar{X}) dt + \sigma_x dZ_2
\]
Model

- The investor’s utility

\[ U(W) = \frac{W^{1-\gamma}}{1-\gamma} \]

- Insurance premium

\[ P_t = \mu_t (Z_t - W_t) \]
Model

- Budget constraint

\[ dW = rWdt + \left( \frac{dS}{S} - r \right) \theta W dt - c_t dt - P_t dt \]
The objective of the investor is to maximise utility. This is mathematically given by:

\[ J(W, X_t, t) = \max_{\theta, C} E_t \left\{ \int_t^\omega p_t \mu_{t+T} \left[ \int_t^T U(C_s)ds + U(Z_T) \right] dT \right\} \]
Numerical method

• A numeric approach was used to solve this problem and the Japanese economy was used as parameters.
Results

Optimal Allocation

- Total No Jump
- Myopic No Jump
- Total Jump
- Myopic Jump

Allocation to Risky Asset (%) vs. Age

- 0.00%
- 5.00%
- 10.00%
- 15.00%
- 20.00%
- 25.00%
- 30.00%
- 35.00%
- 40.00%
## Results: Consumption

<table>
<thead>
<tr>
<th>Age</th>
<th>Myopic</th>
<th>Overall</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$c_{ms}$</td>
<td>$c_m$</td>
</tr>
<tr>
<td>30</td>
<td>0.1781</td>
<td>0.1746</td>
</tr>
<tr>
<td>40</td>
<td>0.2022</td>
<td>0.1989</td>
</tr>
<tr>
<td>50</td>
<td>0.2357</td>
<td>0.2326</td>
</tr>
<tr>
<td>60</td>
<td>0.2841</td>
<td>0.2813</td>
</tr>
<tr>
<td>70</td>
<td>0.3583</td>
<td>0.3558</td>
</tr>
<tr>
<td>80</td>
<td>0.4836</td>
<td>0.4817</td>
</tr>
<tr>
<td>90</td>
<td>0.7253</td>
<td>0.7245</td>
</tr>
</tbody>
</table>
Conclusion

• Investors will exploit the correlation between the risky asset and the risk premium to maximise their consumption utility.