Modelling Dependence in Insurance Claims Processes with Lévy Copulas

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based on a paper with

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Acknowledgements

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Outline

1. Motivation
2. Introduction to Lévy Copulas
3. Fitting the Clayton Lévy copula to real data
4. New Lévy copula models
5. Comparison of Lévy copula fit
6. Multivariate Lévy copulas
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Motivation

- A single event may give rise to claims in multiple classes of business due to dependence between classes of business.

- Such dependence has implications for diversification benefits and in turn impacts:
  - Capital adequacy;
  - Capital allocation;
  - Pricing.

- Distributional copulas can be used to model dependence between aggregate claim amounts in multiple classes of business.
Complications with distributional copulas

- Suppose at time $T$, aggregate claims amounts in $d$ classes of business are described by probability distribution functions $F_{T,1}(x_1), \ldots, F_{T,d}(x_d)$.
- Copulas can be used to construct a multivariate distribution of aggregate claims:
  \[
  F_T(x_1, \ldots, x_d) = C_T(F_{T,1}(x_1), \ldots, F_{T,d}(x_d)).
  \] (1)
- The copula $C_T$ will generally depend on $T$.
- The copula $C_S$ at time $S$ cannot be derived from $C_T$.
- Dependence in frequency and dependence in severity cannot be modelled separately.
Advantages of Lévy Copulas

- Lévy copulas have advantages over distributional copulas:
  - Dependence in claim frequency separate to dependence in claim severity;
  - Time-consistent - allowing for simple changes of risk horizon;
  - Parsimonious.

- Towers Watson survey (Fosker et al., 2010) indicates insurers may prefer different risk horizons in economic capital considerations. E.g. one-year, five-year or runoff of the portfolio.
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Compound Poisson process

- The compound Poisson process presents itself as a natural choice for modelling aggregate claim amounts over continuous time:

\[ S(t) = \sum_{i=1}^{N(t)} X_i, \]  

- The Poisson process \( N(t) \) is the number of claims up to time \( t \) and has parameter \( \lambda \), the expected number of claims per unit of time.
- Claim severities \( X_1, \ldots, X_{N(t)} \) are identically, independently distributed with survival function \( \overline{F}(x) = \Pr(X > x) \).
Two classes of business
Two classes of business
A bivariate compound Poisson process \((S_1(t), S_2(t))\) can be broken up into independent parts and (jump) dependent parts:

\[
S_1(t) = S_1^\perp(t) + S_1^\parallel(t)
\]
\[
S_2(t) = S_2^\perp(t) + S_2^\parallel(t)
\]

\(S_1^\parallel(t)\) and \(S_2^\parallel(t)\) represent common claims with expected frequency \(\lambda^\parallel\).

The sizes of common claims may be dependent.

The joint survival function of the common jump sizes is denoted by \(\bar{F}^\parallel(x_1, x_2)\).

Fitting all components separately is not parsimonious.
For a class of business $i$, the tail integral $U_i(x)$ is the expected number of claims of size greater than $x$:

$$U_i(x) = \lambda_i \bar{F}_i(x), \quad (4)$$
$$= \lambda_i \Pr(X > x). \quad (5)$$
Tail integral

\[ U(x) = \lambda \Pr(X > x) \]

(Expected no. of claims with size > x)
Lévy Copulas

- The joint tail integral measures jumps which occur simultaneously and is given by

\[ U(x_1, x_2) = \lambda \| F \| (x_1, x_2). \]  

(6)

- A Lévy copula \( \mathcal{C} \), couples the marginal tail integrals and the joint tail integral:

\[ \mathcal{C}(U_1(x_1), U_2(x_2)) = U(x_1, x_2) \]  

(7)
Dependence in frequency and severity

- A Lévy copula parsimoniously creates dependence between two compound Poisson processes in terms of frequency and severity.

\[ \lambda^\parallel = \mathcal{C}(\lambda_1, \lambda_2) \]  
\[ \overline{F}^\parallel(x_1, x_2) = \frac{1}{\lambda^\parallel} \mathcal{C}(\lambda_1 \overline{F}_1(x_1), \lambda_2 \overline{F}_2(x_1)). \]
Distribution of claim sizes

- In general, unique jump sizes are distributed differently to common jump sizes.

\[ \lambda_i^{-} = \lambda_i - \lambda_{||} \]  
\[ F_i^{-}(x) = \frac{1}{\lambda_i^{-}} \left( \lambda_i F_i(x) - \lambda_{||} F_{||}(x) \right). \]

- Example - Clayton Lévy copula:

\[ C_\delta(u_1, u_2) = \left( u_1^{-\delta} + u_2^{-\delta} \right)^{-\frac{1}{\delta}}. \]

- Survival copula of common jump sizes is a Clayton distributional copula.
Existing literature

- Bregman and Klüppelberg (2005) derive ruin probabilities for a company with two classes of business, dependent with a Clayton Lévy copula.
- Böcker and Klüppelberg (2008) derive an approximation for VaR for operational risk where multiple event types are dependent with a Clayton Lévy copula.
- Esmaeili and Klüppelberg (2010) fit a bivariate compound Poisson process to Danish fire insurance data using maximum likelihood estimation and the Clayton Lévy copula to model dependence.
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We use data provided by SUVA, a Swiss worker’s compensation company incorporated under public law.

The dataset consists of a random sample of 5% of claims from the construction sector for accidents incurred in 1999 (developed as at 2003).

It features two types of payments: 2249 medical payments and 1099 daily allowance payments.

1089 claims have payments common to both types.

Compound Poisson processes $S_1(t)$ and $S_2(t)$, with dependence characterised by a Lévy copula, are fit to the SUVA dataset.

We use the Gumbel and Normal distributions for the log of payment sizes in the two types.
Dependence between common claim sizes

Scatterplot of log sizes (left) and empirical copula (right) for 1089 common claim sizes:
Maximum likelihood estimates

<table>
<thead>
<tr>
<th>ln L</th>
<th>$\hat{d}$</th>
<th>$\hat{\lambda}_1$</th>
<th>$\hat{\lambda}_2$</th>
<th>$\hat{\lambda}^*$</th>
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<tbody>
<tr>
<td>8536.43</td>
<td>2.2632 (0.0688161)</td>
<td>2176.90 (46.26)</td>
<td>1066.27 (31.68)</td>
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<tr>
<th>Log Medical (Gumbel)</th>
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<td>$\hat{a}$</td>
<td>$\hat{b}$</td>
</tr>
<tr>
<td>5.1007 (0.0253)</td>
<td>1.1404 (0.0189)</td>
</tr>
</tbody>
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Maximum likelihood estimates

| ln L     | $\hat{\delta}$ | $\hat{\lambda}_1$ | $\hat{\lambda}_2$ | $\hat{\lambda}^{||}$ |
|----------|-----------------|---------------------|---------------------|------------------------|
| 8536.43  | 2.2632          | 2176.90             | 1066.27             | 984.16                 |
|          | (0.0688161)     | (46.26)             | (31.68)             |                        |
| Empirical|                 | 2249                | 1099                | 1089                   |

Log Medical (Gumbel) Log Daily Allowance (Normal)

<table>
<thead>
<tr>
<th>$\hat{a}$</th>
<th>$\hat{b}$</th>
<th>$\hat{\mu}$</th>
<th>$\hat{\sigma}$</th>
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<td>(0.0253)</td>
<td>(0.0189)</td>
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## Maximum likelihood estimates

| ln L     | $\hat{\delta}$  | $\hat{\lambda}_1$ | $\hat{\lambda}_2$ | $\hat{\lambda}^{||}$ |
|----------|------------------|--------------------|---------------------|-----------------------|
| 8536.43  | 2.2632           | 2176.90            | 1066.27             | 984.16                |
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<td>Marginal:</td>
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<tr>
<td>5.1476</td>
<td>1.1048</td>
<td>7.6305</td>
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We can compare the theoretical tail integrals of unique and common jumps for each class with the observed number of claims above size $x$. 

**Quality of fit**

- Medical (Unique)
- Medical (Common)
- Allowance (Common)
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A Lévy copula which induces common jumps with independent sizes and identically distributed unique jump sizes and common jump sizes in each class is the Pure Common Shock (PCS) Lévy copula.

\[ \mathcal{C}(u_1, u_2; \delta) = \delta u_1 u_2 I_{\{u_1 \vee u_2 \neq \infty\}} + u_1 I_{\{u_1 = \infty\}} + u_2 I_{\{u_2 = \infty\}}, \]

for \( \delta \leq \min\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}\right) \)

This produces a bivariate compound Poisson process with dependence in frequency only.
Archimedean Lévy copula

- Tankov (2003) shows that for some strictly decreasing and convex function $\phi$,
  \[ c(u_1, u_2) = \phi^{-1} \left( \phi(u_1) + \phi(u_2) \right), \tag{13} \]
defines a bivariate Archimedean Lévy copula (easily extended to multivariate).
- We develop 2 new Archimedean models and compare them through analysis of the Lévy copula density as well as the dependence structure of the sizes of common jumps.
- We also compare properties under change of time horizon and properties related to dependence in frequency.
Lévy copula density

- Shows the relative prevalence of common jumps at different sizes in each component.
- Its volume reflects the dependence in frequency.
- Example: Consider a bivariate compound Poisson process with $\lambda_1 = 100$, $\lambda_2 = 100$ and a Lévy copula parameter such that $\lambda^\parallel = 60$. 
Pure common shock Lévy copula
Clayton Lévy copula
Archimedean model I
Archimedean model II
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New maximum likelihood estimates

| Lévy copula | ln L   | $\hat{\delta}$ | $\hat{\lambda}_1$ | $\hat{\lambda}_2$ | $\hat{\lambda}_{||}$ |
|------------|--------|-----------------|-------------------|-------------------|---------------------|
| AM1        | 8631.27| 0.0025358       | 2239.42           | 1113.32           | 1093.74             |
| Clayton    | 8536.43| 2.2632          | 2176.90           | 1066.27           | 984.16              |
| PCS        | 7845.03| 0.0004406       | 2249.00           | 1099.00           | 1089.00             |
New maximum likelihood estimates

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Pure common shock Lévy copula

Medical (Unique)  Medical (Common)  Allowance (Common)
Archimedean model I

Medical (Unique)  Medical (Common)  Allowance (Common)

A significant improvement over the Clayton Lévy copula!
Dependence in common claim sizes

- Scatterplots of copula simulations for common jump sizes:

Clayton

Archimedean model I
Comparison to empirical copula
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Three classes of business

For the case of three dependent classes of business:

\[
\begin{align*}
S_1(t) &= S_{11}(t) + S_{112}(t) + S_{113}(t) + S_{1123}(t), \\
S_2(t) &= S_{21}(t) + S_{212}(t) + S_{213}(t) + S_{2123}(t), \\
S_3(t) &= S_{31}(t) + S_{312}(t) + S_{313}(t) + S_{3123}(t),
\end{align*}
\]

(14)

- Compound Poisson processes $S_{i;ij}(t)$ feature an arrival process common with compound Poisson processes $S_{j;ij}(t)$.
- The three compound Poisson processes denoted as $S_{i;123}(t)$ all feature a common arrival process.
Summary

Contributions:

- New Lévy copula models, including the Pure Common Shock Lévy copula, allowing for dependence in frequency only.
- Fit Lévy copulas to a new set of real data.
- Illustrated a way of comparing the fit of different Lévy copulas.
- Derived a common shock relationship for a trivariate Lévy copula model (trivariate formulation)
Thank you

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References (cont.)
