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Construction of detailed correlation structures across GI business segments

Benjamin Avanzi, Greg Taylor & Bernard Wong

School of Risk and Actuarial Studies
UNSW Australia

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The authors and their Linkage Project

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  - Term: 3 years+
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Overview

- Motivation
- Common shock models
- Application to multiple claim triangles
- Reduction to simple concepts for populating large correlation matrices
- Numerical example for risk margins
- Capital margins
- Conclusion

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The "why should we care?" test (1)

- Who cares about large correlation matrices?
- **Risk margins** (moderate percentiles of **total** liabilities)
- **Capital margins** (high percentiles of **total** liabilities)
  - Both require consideration of dependencies between **business segments**
  - For some purposes, the dependencies may be expressed as correlations

The “why should we care?” test (2)

- A large insurer may wish to recognize 50 or more segments
- Simple case
  - Only 50 segments
  - No fine structure within segments
  - 50x50 correlation matrix has **1,225** free entries requiring estimation

The “why should we care?” test (3)

- It gets worse
- A claim triangle may be associated with each business segment
- A dependency between two segments may differ according to the cells of the triangles considered
  - e.g. suppose correlation exists specifically between diagonals

The “why should we care?” test (4)

- Simple case
  - Only 50 segments
  - Triangles only 10x10
    - 45 cells each in lower triangle (projected future)
  - There are now 2,250 cells
  - 2250x2250 correlation matrix has roughly **2.5M** free entries requiring estimation

The "why should we care?" test (5)

- So how should one proceed with the generation of these large matrices and be certain of satisfying the following requirements:
  - Matrix is known to be positive definite
  - The magnitude of each entry is reasonable
  - The relative magnitudes of any pair of entries are reasonable
    - Note that, in our simple example, there are roughly 3 trillion pairs

Scope

- We shall mainly discuss correlations
- These give meaningful representations of dependency only for distributions that do not deviate too far from normal
- They are therefore suitable for measurement of insurance claim dependencies not too distant from the mean (moderate percentiles)
- Most of the presentation therefore relates to **risk margins** rather than capital margins
- But a brief word about capital margins at the end

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A simple (trivial?) model

- Suppose X, Y, Z independent random variables
- Define new variables

$$A = \alpha_A Z + \beta_A X$$
$$B = \alpha_B Z + \beta_B Y$$

where α's, β's are constants > 0

- Evidently, A, B are dependent provided Z is not degenerate
- In fact

$$Cov(A, B) = \alpha_A \alpha_B \sigma_Z^2 \geq 0$$

- This is a **common shock model**
  - It forms the basis of almost the entire presentation

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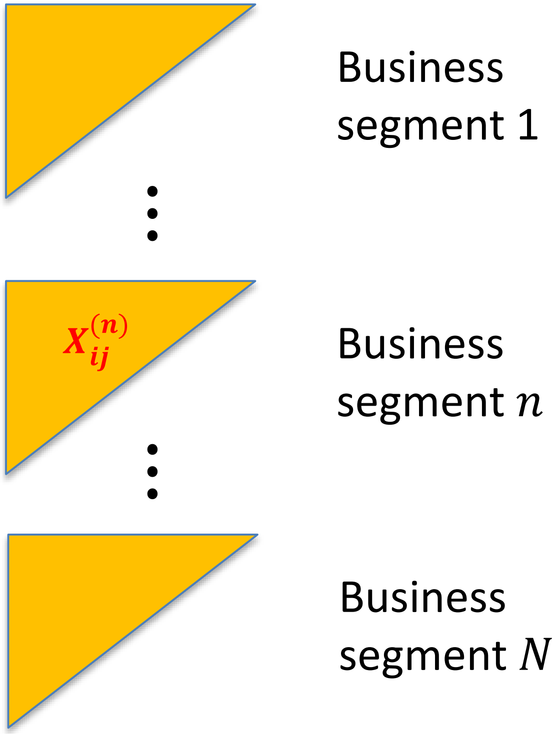
Framework and notation (1)

- Consider N business segments
- Each associated with an (upper) claim triangle with entries X labelled by accident and development period
- So $\boldsymbol{X_{ij}^{(n)}}$ denotes the entry (e.g. claim payments) in segment n for accident period i and development period j
- All triangles congruent (same size and shape)
- We could relax these conditions if we wished
  - Don't require triangles, only 2-D arrays of any shape, possibly with holes
  - Don't require congruence

Framework and notation (2)

- So framework has this appearance



Construction of dependent models

- Dependency might occur:
  - Within a single triangle; or
  - Between a number of different triangles;
  - Or both

Within-triangle dependency (1)

- Suppose one wishes to create dependency between cells (i, j) and (k, ℓ) of triangle n
- Just define the common shock model (for all i, j)

$$X_{ij}^{(n)} = \underbrace{\beta_{ij}^{(n)} W^{(n)}}_{\text{Common shock component}} + \underbrace{\phi_{ij}^{(n)} Z_{ij}^{(n)}}_{\text{Idiosyncratic component}}$$

$W^{(n)}$ and all $Z_{ij}^{(n)}$ independent

Within-triangle dependency (2)

$$X_{ij}^{(n)} = \beta_{ij}^{(n)} W^{(n)} + \phi_{ij}^{(n)} Z_{ij}^{(n)} \left(\boldsymbol{\beta_{ij}^{(n)} > 0}\right)$$

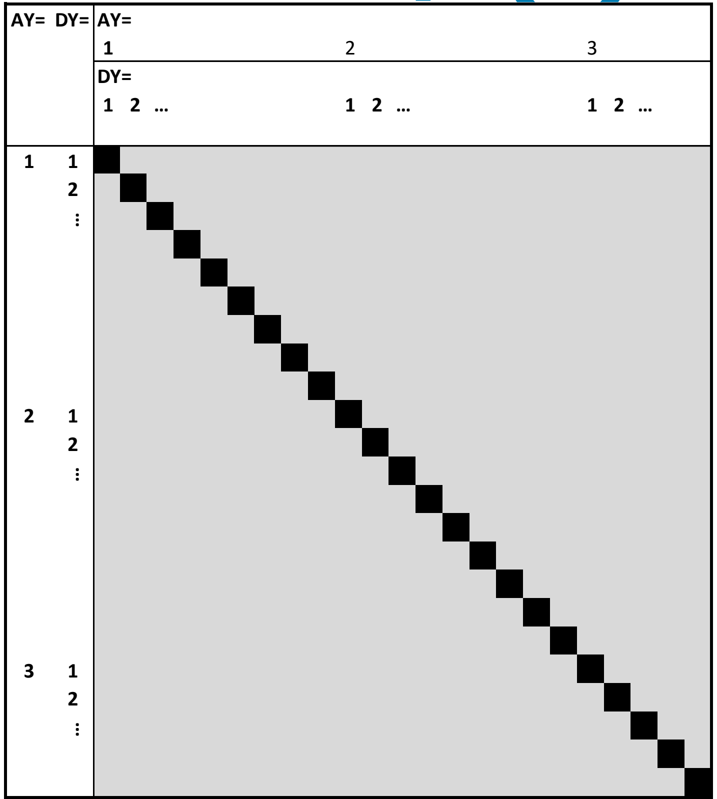
- It follows that

$$Cov\left[X_{ij}^{(n)}, X_{k\ell}^{(n)}\right] = \beta_{ij}^{(n)} \beta_{k\ell}^{(n)} \sigma_{W^{(n)}}^2 + \delta_{ij,k\ell} \left(\phi_{ij}^{(n)}\right)^2 \sigma_{Z_{ij}^{(n)}}^2 > \mathbf{0}$$

- Note that this creates dependency between all cells of triangle n
- Note also that the matrix of covariances is strictly positive definite by construction
  - Same comment for all dependencies considered henceforth

Within-triangle dependency (3)

- The covariance matrix takes the schematic form illustrated
  - Axes labelled by DY within AY
  - Darker shading indicates greater covariance



Row-wise dependency (1)

$$X_{ij}^{(n)} = \beta_{ij}^{(n)} W^{(n)} + \phi_{ij}^{(n)} Z_{ij}^{(n)}$$

$$Cov\left[X_{ij}^{(n)}, X_{k\ell}^{(n)}\right] = \beta_{ij}^{(n)} \beta_{k\ell}^{(n)} \sigma_{W^{(n)}}^2 + \delta_{ij,k\ell} \left(\phi_{ij}^{(n)}\right)^2 \sigma_{Z_{ij}^{(n)}}^2$$

- Suppose one wishes to introduce only a row-wise dependency, i.e.

$$Cov\left[X_{ij}^{(n)}, X_{k\ell}^{(n)}\right] > 0 \; iff \; i = k$$

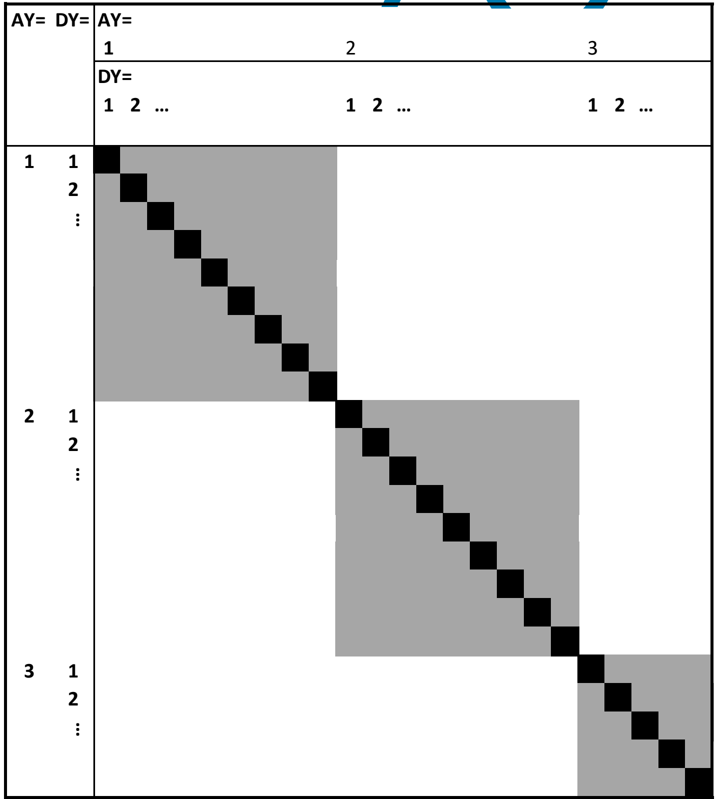
- Then simply replace the model by:

$$X_{ij}^{(n)} = \beta_{ij}^{(n)} \boldsymbol{W_i^{(n)}} + \phi_{ij}^{(n)} Z_{ij}^{(n)}$$

$$Cov\left[X_{ij}^{(n)}, X_{k\ell}^{(n)}\right] = \boldsymbol{\delta_{ik}} \beta_{ij}^{(n)} \beta_{k\ell}^{(n)} \boldsymbol{\sigma_{W_i^{(n)}}^2} + \delta_{ij,k\ell} \left(\phi_{ij}^{(n)}\right)^2 \sigma_{Z_{ij}^{(n)}}^2$$

Row-wise dependency (2)

- Covariance matrix



Column- and diagonal-wise dependency

- Row-wise

$$X_{ij}^{(n)} = \beta_{ij}^{(n)} W_i^{(n)} + \phi_{ij}^{(n)} Z_{ij}^{(n)}$$

- Column-wise

$$X_{ij}^{(n)} = \beta_{ij}^{(n)} W_{\boldsymbol{j}}^{(n)} + \phi_{ij}^{(n)} Z_{ij}^{(n)}$$

- Diagonal-wise

$$X_{ij}^{(n)} = \beta_{ij}^{(n)} W_{\boldsymbol{t}}^{(n)} + \phi_{ij}^{(n)} Z_{ij}^{(n)} \text{ where } \boldsymbol{t = i + j - 1}$$

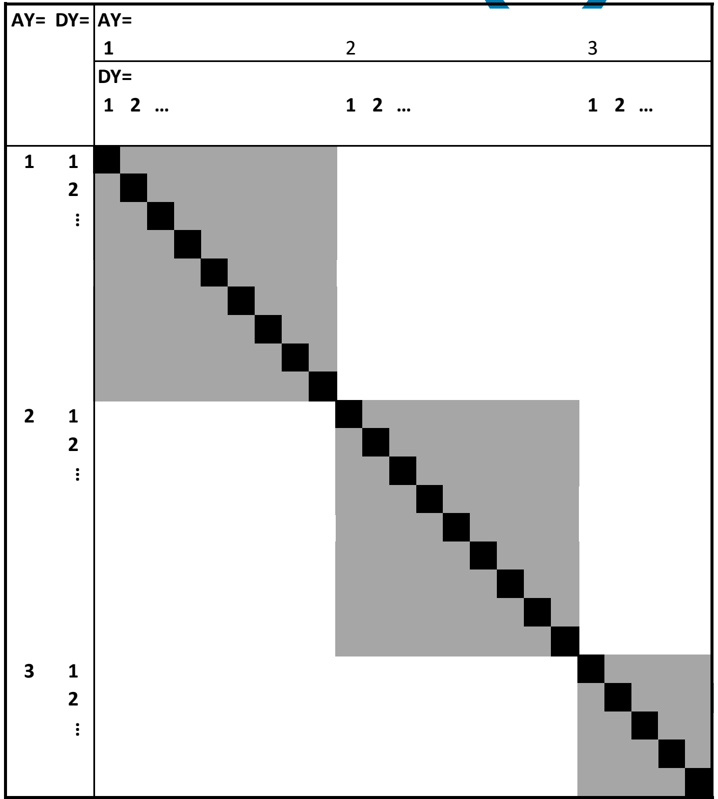
More general within-array dependencies

- All of the previous forms of dependency can be present simultaneously

$$X_{ij}^{(n)} = \beta_{(arr)ij}^{(n)} W_{(arr)}^{(n)} + \beta_{(row)ij}^{(n)} W_{(row)i}^{(n)} + \beta_{(col)ij}^{(n)} W_{(col)j}^{(n)} + \beta_{(diag)ij}^{(n)} W_{(diag)t}^{(n)} + \phi_{ij}^{(n)} Z_{ij}^{(n)}$$

Time series dependencies (1)

- Re-consider the row dependency introduced (see right)
- Dependency occurs only within rows
- Observations from different rows are independent
- One may desire a more graded approach
  - All rows are dependent, but
  - Dependency decreases with increasing distance between rows



Time series dependencies (2)

- Consider an AR(1) time series

$$D_t = \theta D_{t-1} + \varepsilon_t, E[\varepsilon_t] = 0, Var[\varepsilon_t] = \sigma_\varepsilon^2$$

- May be shown that

$$Cov[D_s, D_t] \cong const. \times \theta^{t-s}, t > s$$

for s, t sufficiently large for the series to have "forgotten" its initial value

- This kind of geometric decay $(0 \leq \theta \leq 1)$may be more suitable for correlation between rows (or columns, or diagonals)

Time series dependencies (3)

- Example: dependency between diagonals
- Earlier form of diagonal-wise dependency model:

$$X_{ij}^{(n)} = \beta_{ij}^{(n)} W_t^{(n)} + \phi_{ij}^{(n)} Z_{ij}^{(n)}, all\ W_t^{(n)}\ indep.$$

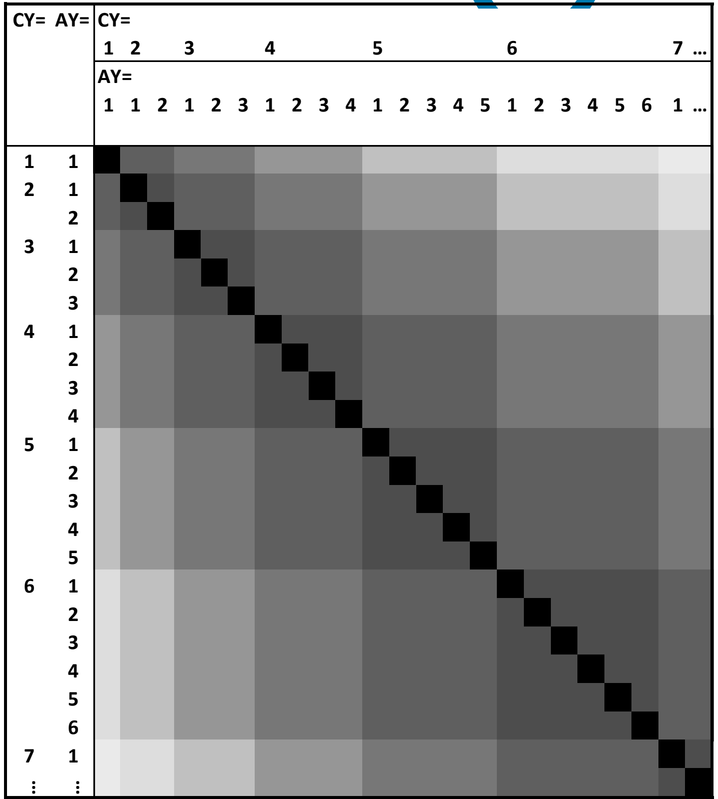
- Retain this model form but now assume the $W_t^{(n)}$ are AR(1):
  $W_t^{(n)} = \theta W_{t-1}^{(n)} + \varepsilon_t$
- Then

$$Cov\left[X_{ij}^{(n)}, X_{k\ell}^{(n)}\right] = const. \times \beta_{ij}^{(n)} \beta_{k\ell}^{(n)} \theta^{|(i+j)-(k+\ell)|} + \delta_{ij,k\ell} \left(\phi_{ij}^{(n)}\right)^2 \sigma_{Z_{ij}^{(n)}}^2$$

Distance between diagonals

Time series dependencies (4)

- The covariance matrix now takes the schematic form illustrated
  - Axes now conveniently labelled by AY within CY
- Much richer covariance structure



Between-triangle dependencies (1)

- Suppose one wishes to reflect dependency between cells of different triangles, i.e. between $X_{ij}^{(m)}, X_{k\ell}^{(n)}$
- Consider diagonal-wise dependency as a (more or less arbitrary) example for explanatory purposes

Between-triangle dependencies (2)

- To incorporate diagonal-wise dependency **within** a triangle:

$$X_{ij}^{(n)} = \beta_{ij}^{(n)} W_t^{(n)} + \phi_{ij}^{(n)} Z_{ij}^{(n)}$$

$$Cov\left[X_{ij}^{(n)}, X_{k\ell}^{(n)}\right] = \delta_{i+j,k+\ell}\beta_{ij}^{(n)}\beta_{k\ell}^{(n)}\sigma_{W_t^{(n)}}^2 + \delta_{ij,k\ell}\left(\phi_{ij}^{(n)}\right)^2\sigma_{Z_{ij}^{(n)}}^2$$

- To add diagonal-wise dependency **between** triangles:

$$X_{ij}^{(n)} = \alpha_{ij}^{(n)} W_t + \beta_{ij}^{(n)} W_t^{(n)} + \phi_{ij}^{(n)} Z_{ij}^{(n)}$$

$$Cov\left[X_{ij}^{(m)}, X_{k\ell}^{(n)}\right]$$

$$= \delta_{i+j,k+\ell}\alpha_{ij}^{(m)}\alpha_{k\ell}^{(n)}\sigma_{W_t}^2 + \delta_{mn}\left[\delta_{i+j,k+\ell}\beta_{ij}^{(n)}\beta_{k\ell}^{(n)}\sigma_{W_t^{(n)}}^2 + \delta_{ij,k\ell}\left(\phi_{ij}^{(n)}\right)^2\sigma_{Z_{ij}^{(n)}}^2\right]$$

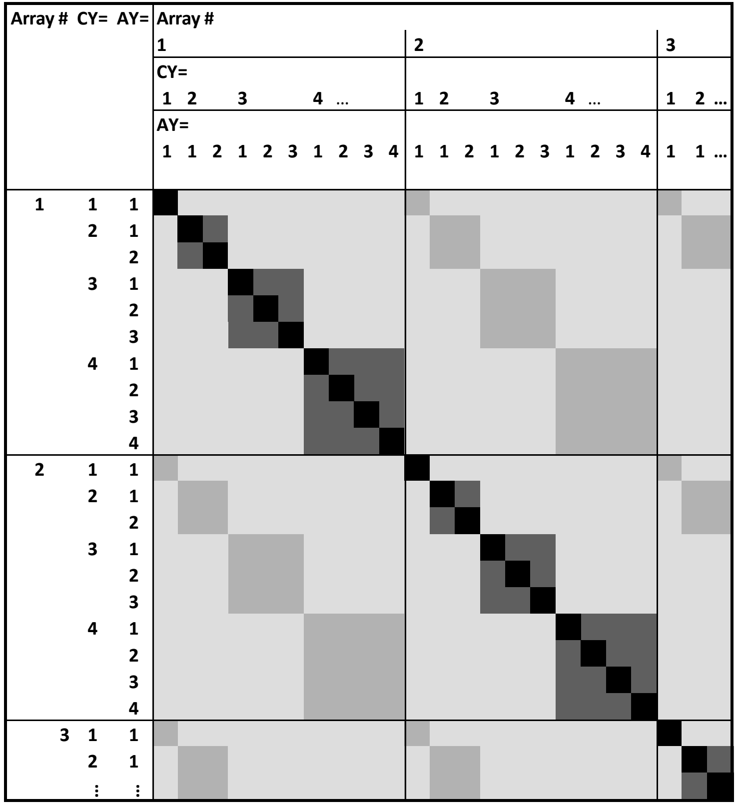
Between triangles, diagonal

Within triangle, diagonal

Within triangle, cell variance

Between-triangle dependencies (3)

- The multi-segment covariance matrix takes the schematic form illustrated
  - Axes now labelled by AY within CY within segment
- Other between-triangle dependencies can be added in similar fashion



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Parameter estimation (1)

- Hitherto, we have been adding common shock terms willy-nilly into the representation of $X_{ij}^{(n)}$, without any thought for how the model is to be implemented
- Ideal if these terms could be formally estimated
  - Some literature on this
- But, in many practical situations, estimation will be heuristic (translation: informed guesswork)
  - Particularly the case for forecasting (reserving)
    - [Niels Bohr: “Prediction is very difficult, especially if it’s about the future”]

Parameter estimation (2)

- So we now concentrate on reducing the results to a form that:
  - Is brief and palatable
  - Consists of terms that are:
    - Relatively few in number
    - Intuitive in their interpretation
- But without major loss of accuracy
- This will provide the practitioner with a reasonable chance of reasonable accuracy in heuristic parameter estimation

Parameter reduction (1)

- As an example, recall the between-triangle diagonal dependency case

$$Cov\left[X_{ij}^{(m)}, X_{k\ell}^{(n)}\right]$$

$$= \delta_{i+j,k+\ell}\alpha_{ij}^{(m)}\alpha_{k\ell}^{(n)}\sigma_{W_t}^2 + \delta_{mn}\left[\delta_{i+j,k+\ell}\beta_{ij}^{(n)}\beta_{k\ell}^{(n)}\sigma_{W_t^{(n)}}^2 + \delta_{ij,k\ell}\left(\phi_{ij}^{(n)}\right)^2\sigma_{Z_{ij}^{(n)}}^2\right]$$

- The first simplification arises from noting that the σ terms can all be absorbed into their associated coefficients:

$$Cov\left[X_{ij}^{(m)}, X_{k\ell}^{(n)}\right]$$

$$= \delta_{i+j,k+\ell}\alpha_{ij}^{(m)}\alpha_{k\ell}^{(n)} + \delta_{mn}\left[\delta_{i+j,k+\ell}\beta_{ij}^{(n)}\beta_{k\ell}^{(n)} + \delta_{ij,k\ell}\left(\phi_{ij}^{(n)}\right)^2\right]$$

Parameter reduction (2)

$$Cov\left[X_{ij}^{(m)}, X_{k\ell}^{(n)}\right] = \delta_{i+j,k+\ell}\alpha_{ij}^{(m)}\alpha_{k\ell}^{(n)} + \delta_{mn}\left[\delta_{i+j,k+\ell}\beta_{ij}^{(n)}\beta_{k\ell}^{(n)} + \delta_{ij,k\ell}\left(\phi_{ij}^{(n)}\right)^2\right]$$

- Special case: $(m, i, j) = (n, k, \ell)$

$$Var\left[X_{ij}^{(m)}\right] = \left(\alpha_{ij}^{(m)}\right)^2 + \left(\beta_{ij}^{(m)}\right)^2 + \left(\phi_{ij}^{(m)}\right)^2$$

- The nature of these three components was noted earlier
- So cell variance decomposes into contributions from:
  - Diagonal common shock across all triangles
  - Diagonal common shock specific to the triangle
  - Idiosyncratic noise specific to cell
- A variance decomposition for each cell determines all coefficients of the dependency structure (apart from θ's if they are included)
  - θ's would be estimated/guesstimated separately

Parameter reduction (3)

- Further mathematical development is omitted
- Proceeding directly to the conclusion, the entire dependency structure is defined by the following parameters
  - For each cell in each triangle
    - The decomposition of the cell variance into its three components
  - For each triangle
    - The value of the AR(1) coefficient if time series effects are included
  - Across all triangles
    - The value of the AR(1) coefficient if a time series common shock across all triangles is included
- A total of $\mathbf{3N+1}$ parameter values to be specified

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Numerical example

- Need to consider small dimensions in order that results may be displayed
- Choose $N = 2, I = J = 4$ (4x4 triangles)
  - 10 observations per triangle
  - 20x20 correlation matrix

Selection of parameter values

- For each cell in each triangle
  - The decomposition of the cell variance into its three components
    - **Same for all cells in a triangle**
    - **Triangle 1: 0.1, 0.3, 0.6**
    - **Triangle 2: 0.1, 0.1, 0.8**
- For each triangle
  - The value of the AR(1) coefficient if time series effects are included
    - **Triangles 1 and 2: 0.3, 0.6**
- Across all triangles
  - The value of the AR(1) coefficient if a time series common shock across all triangles is included: **0.2**
- Correlation matrix follows very quickly and easily

Example correlation matrix

- Within-diagonal covariances indicated by shading

Class #	CY=	AY=	Class #																			
			1										2									
			CY=																			
			1	2	2	3	3	3	4	4	4	4	1	2	2	3	3	3	4	4	4	4
			AY=																			
			1	1	2	1	2	3	1	2	3	4	1	1	2	1	2	3	1	2	3	4
1	1	1	1.00	0.11	0.11	0.03	0.03	0.03	0.01	0.01	0.01	0.01	0.10	0.02	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	2	1	0.11	1.00	0.40	0.11	0.11	0.11	0.03	0.03	0.03	0.03	0.02	0.10	0.10	0.02	0.02	0.02	0.00	0.00	0.00	0.00
	2	2	0.11	0.40	1.00	0.11	0.11	0.11	0.03	0.03	0.03	0.03	0.02	0.10	0.10	0.02	0.02	0.02	0.00	0.00	0.00	0.00
	3	1	0.03	0.11	0.11	1.00	0.40	0.40	0.11	0.11	0.11	0.11	0.00	0.02	0.02	0.10	0.10	0.10	0.02	0.02	0.02	0.02
	3	2	0.03	0.11	0.11	0.40	1.00	0.40	0.11	0.11	0.11	0.11	0.00	0.02	0.02	0.10	0.10	0.10	0.02	0.02	0.02	0.02
	3	3	0.03	0.11	0.11	0.40	0.40	1.00	0.11	0.11	0.11	0.11	0.00	0.02	0.02	0.10	0.10	0.10	0.02	0.02	0.02	0.02
	4	1	0.01	0.03	0.03	0.11	0.11	0.11	1.00	0.40	0.40	0.40	0.00	0.00	0.00	0.02	0.02	0.02	0.10	0.10	0.10	0.10
	4	2	0.01	0.03	0.03	0.11	0.11	0.11	0.40	1.00	0.40	0.40	0.00	0.00	0.00	0.02	0.02	0.02	0.10	0.10	0.10	0.10
	4	3	0.01	0.03	0.03	0.11	0.11	0.11	0.40	0.40	1.00	0.40	0.00	0.00	0.00	0.02	0.02	0.02	0.10	0.10	0.10	0.10
	4	4	0.01	0.03	0.03	0.11	0.11	0.11	0.40	0.40	0.40	1.00	0.00	0.00	0.00	0.02	0.02	0.02	0.10	0.10	0.10	0.10
2	1	1	0.10	0.02	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.08	0.08	0.04	0.04	0.04	0.02	0.02	0.02	0.02
	2	1	0.02	0.10	0.10	0.02	0.02	0.02	0.00	0.00	0.00	0.00	0.08	1.00	0.20	0.08	0.08	0.08	0.04	0.04	0.04	0.04
	2	2	0.02	0.10	0.10	0.02	0.02	0.02	0.00	0.00	0.00	0.00	0.08	0.20	1.00	0.08	0.08	0.08	0.04	0.04	0.04	0.04
	3	1	0.00	0.02	0.02	0.10	0.10	0.10	0.02	0.02	0.02	0.02	0.04	0.08	0.08	1.00	0.20	0.20	0.08	0.08	0.08	0.08
	3	2	0.00	0.02	0.02	0.10	0.10	0.10	0.02	0.02	0.02	0.02	0.04	0.08	0.08	0.20	1.00	0.20	0.08	0.08	0.08	0.08
	3	3	0.00	0.02	0.02	0.10	0.10	0.10	0.02	0.02	0.02	0.02	0.04	0.08	0.08	0.20	0.20	1.00	0.08	0.08	0.08	0.08
	4	1	0.00	0.00	0.00	0.02	0.02	0.02	0.10	0.10	0.10	0.10	0.02	0.04	0.04	0.08	0.08	0.08	1.00	0.20	0.20	0.20
	4	2	0.00	0.00	0.00	0.02	0.02	0.02	0.10	0.10	0.10	0.10	0.02	0.04	0.04	0.08	0.08	0.08	0.20	1.00	0.20	0.20
	4	3	0.00	0.00	0.00	0.02	0.02	0.02	0.10	0.10	0.10	0.10	0.02	0.04	0.04	0.08	0.08	0.08	0.20	0.20	1.00	0.20
	4	4	0.00	0.00	0.00	0.02	0.02	0.02	0.10	0.10	0.10	0.10	0.02	0.04	0.04	0.08	0.08	0.08	0.20	0.20	0.20	1.00

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Current practice

- Noted earlier that correlations alone are not helpful for estimation of the extreme tail
- Common practice is to combine a t-copula with estimated marginal distributions for business segments
- t-copula defined by correlation matrix and degrees of freedom
  - Often practical difficulty in selecting these
- The present work may be extended slightly to inform the calculation of capital margins

Tail dependency

- The choice of t-copula degrees of freedom may be best approached in terms of the **coefficient of (upper) tail dependency**
  - This is a quantity specific to the extreme tails
  - **Definition:** $\lambda = \lim_{q \to 1^-} Prob[X_2 > F_2^{\leftarrow}(q) | X_1 > F_1^{\leftarrow}(q)]$ where
    - F_i is the d.f. of X_i
    - $F_i^{\leftarrow}$ is the **generalized inverse** of F_i, i.e. $F_i^{\leftarrow}(y) = inf\{x: F_i(x) \geq y\}$
- A capital actuary would normally be able to take a view on the limiting conditional probability involved in the definition of the tail dependency

Selection of t-copula

- If the copula is made subject to the correlation matrix calculated earlier, then it will be consistent with any risk margins calculated
- Its tail behavior will be determined by its degrees of freedom
- So
  - Estimate the coefficient of tail dependency for all pairs of segments
  - Tabulate the coefficients of tail dependency given for these pairs according to a t-copula for varying degrees of freedom
  - Select the number of degrees of freedom that gives a rough match (if a match exists)
  - The resulting copula will be consistent with both risk margins and the actuary's views of tail behavior
- Note that the non-existence of a match indicates that a t-copula is inconsistent with these other criteria

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Conclusion

- Dependency models constructed across triangles for multiple business segments
- Flexible models that allow for
  - Within- and between-triangle dependencies
  - Row-, column- and diagonal-wise dependencies (and, indeed, just about anything else)
  - Time series dependencies between different rows, etc.
- Expression of the models in a parametrization that is
  - Frugal in the number of parameters
  - Intuitive in interpretation
- Models applicable directly to risk margins
  - But also applicable to capital margins under a simple extension

Questions?