Estimating Value at Risk of Portfolio:
Skewed-EWMA Forecasting via Copula

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Talk outline

- **Why important**?

-- Background on financial risk modeling.

EWMA: exponentially weighted moving average

- **Why difficult**?

-- stylized facts on financial return series

- **How to solve**?

-- Asymmetric Laplace distribution

-- Skewed-EWMA and forecasting of VaR via copula

-- Evaluation of VaR forecasting

- **Outlook**

I. Why important?

Background on financial risk modeling

- **Financial practice / globalization poses new challenging questions on financial risks**, e.g., modeling / pricing of
  - financial risks (market risk, credit risk, etc) and hedging derivatives
- **Financial innovation requires and produces new financial instruments:** e.g., modeling/pricing of ***securitization*** of insurance risks
  - weather risk -- weather derivatives,
  - catastrophe risk -- catastrophe insurance derivatives

Financial market risk

- ***Risk management*** has truly experienced a ***revolution*** in the last few years. This was started by ***value at risk*** (VaR), a new method to measure financial market risk that was developed in response to the financial disasters of the early 1990s.

  --**Philippe Jorion** (2001, preface): *Value at risk: The new benchmark for managing financial risk*

Examples of financial disasters

- Barrings, UK, 02/1995, loss $1.33 billion
- Metallgesellschaft, Germany, 01/1994, loss $1.34 billion
- Orange county, USA, 12/1994, loss $1.81 billion
- Daiwa, Japan, 09/1995, loss $1.1 billion
- **Asia's 1997 financial market turmoil**

ING BARINGS

Daiwa

Importance of VaR modeling

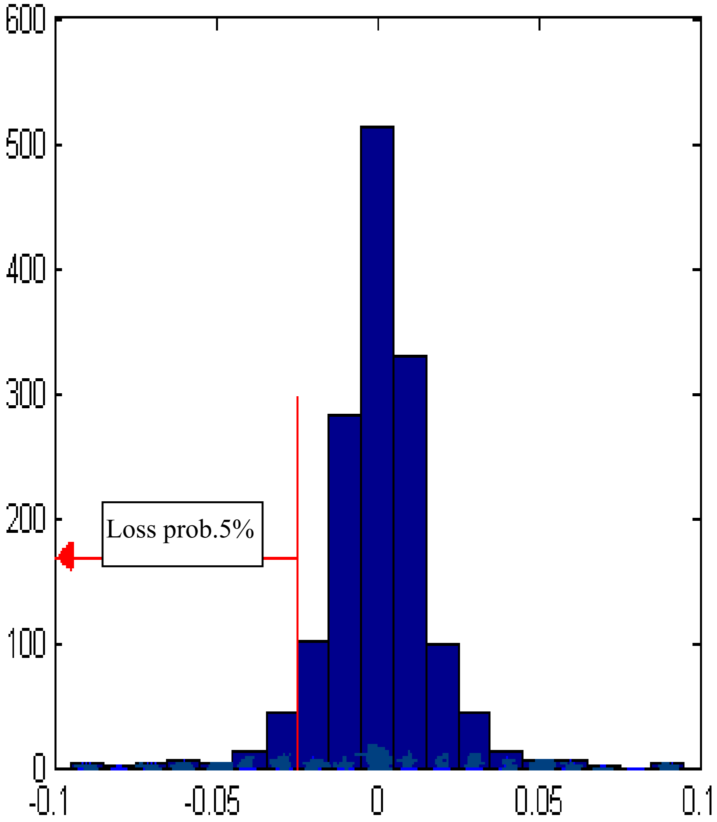
- **Group of Thirty (G-30)** in 1993 advised to value positions using market prices and to *assess financial risks with VaR*.
- Benchmark risk measure for minimum reserve capital, recommended by ***Basel Accord***, the ***U.S. federal reserve bank***, and ***EU's Capital Adequacy Directives, etc***.
- **The greatest advantage**: summarises *in a single, easy to understand number* the downside risk of an institution due to financial market factors.
- Information reporting (passive), controlling risk (defensive), managing risk (active)

II. Why difficult? VaR modeling

- **VaR**: *expected maximum loss* (or worst loss) of a financial variable / portfolio over *a target horizon* at *a given confidence level*, α .
  e.g., α =95%

- **In terms of *return series* of** *the target horizon* , with a distribution F(x), then *the worst return* at *the given confidence level*, α , is:

$$VaR = F^{-1}(1-\alpha).$$



Normal distribution

- Black-Scholes model:

$$dS_t = S_t \alpha \, dt + S_t \sigma \, dB_t$$

B_t - standard Brownian motion, $\sigma > 0$.

- Gemoetric retern:

$$r_t = \log S_t - \log S_{t-1} \sim N(\mu, \sigma^2) : f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right)$$

Where $\mu = \alpha - \sigma^2 / 2$.

- VaR:

VaR= $\mu + \sigma \, \Phi^{-1}(1-\alpha)$,

Φ(.) is c.d.f. of $N(0,1)$. **Practically, assume** μ =0.

Stylized facts of financial return

- Return series show little autocorrelation, but not i.i.d.
- Conditional expected returns are closed to zero
- Absolute or squared return series show profound autocorrelation
- Volatility appears to vary over time
- Extreme returns appear in cluster
- In reality, distribution of return series is *skewed and heavy-tailed & high-peaked*, departure from normality.

-- c.f., Taylor (2005): Asset price dynamics, volatility, and prediction. Princeton University Press.

Changing volatility: Standard-EWMA

- **JP Morgan RiskMetrics: Exponentially weighted moving average (EWMA)**

➢ assume conditional normality for return series， with volatility modeled as IGARCH(1,1) of Engle & Bollersleve (1986),

$$VaR_{t+1} = \sigma_{t+1}\Phi^{-1}(1-\alpha) \quad , \quad \sigma_{t+1}^2 = \lambda\sigma_t^2 + (1-\lambda)r_t^2 \; , \quad 0 < \lambda < 1$$

or equivalently,

$$\sigma_{t+1}^2 = \sum_{i=0}^{\infty} \lambda^i (1-\lambda) r_{t-i}^2 \qquad \Leftarrow \hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T} r_t^2$$

- *geometrically declining weights* on past observations, assigning greater importance to recent observations.

- Nelson and Foster (1994): when returns are conditionally normal， EWMA is optimal.

Changing volatility: Robust-EWMA

- Guermat and Harris (2000): based on *Laplace distribution*

$$r_t \sim LD(\mu,\sigma^2): f(x)=\frac{1}{\sqrt{2}\sigma}\exp\left(-\frac{\sqrt{2}\,|x-\mu|}{\sigma}\right)$$

σ -standard deviation

heavier-tailed than normality.

- Robust-EWMA: **assume** μ =0,

$$VaR_{t+1}=\frac{\sigma_{t+1}}{\sqrt{2}}\ln[2(1-\alpha)]，\quad \sigma_{t+1}=\lambda\sigma_t+(1-\lambda)\sqrt{2}\,|r_t|$$

$$\Leftrightarrow \sigma_{t+1}=\sum_{i=0}^{\infty}\lambda^i(1-\lambda)\sqrt{2}\,|r_{t-i}| \quad \Leftarrow \hat{\sigma}=\frac{1}{T}\sum_{t=1}^{T}\sqrt{2}\,|r_t|$$

- It accounts for heavy tails， but no skewness.
- Absolute return GARCH: Taylor (1986), Schwert (1989)

Importance of skewness

- Documents of skewness of return series, e.g. Kraus and Litzenberger (1976), Friend and Westerfield (1980), Lim(1989), Richardson and Smith(1993), Harvey and Siddique(1999, 2000), Ait-Sahalia and Brandt (2001), Chen(2001)
- Simaan(1993): skewness in portfolio
- Theodossiou(1998): generalized t distribution, too complex

Two challenging questions

How to account for the stylized facts of *skewness* and *heavy tails* simultaneously, which may also change with time, in modeling and forecasting of VaR?

Changing skewness and kurtosis:

Harvey, C. R. and A. Siddique (1999), "Autoregressive Conditional Skewness", *Journal of Financial and Quantitative Analysis* 34, 465-487.

How to account for the complex dependence among individual securities?

III. How to solve?

Our work: Skewed-EWMA via copula

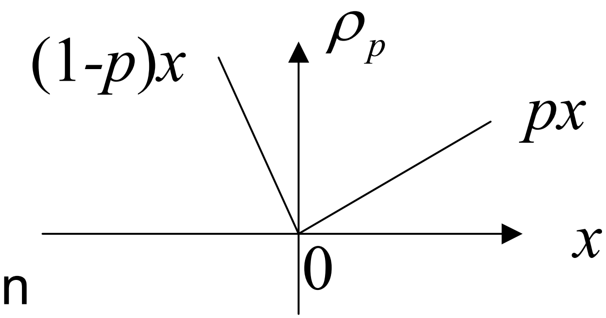
- **A Skewed-EWMA VaR modeling**, based on *Asymmetric Laplace distribution* taking into account both *skewness* and *heavy tails* in financial return series
- **A varying shape parameter by EWMA**, leading to *changing skewness and kurtosis*, adaptive to time-varying nature of financial systems.
- Skewed-EWMA outperforms both Standard- and Robust-EWMAs in VaR forecasting.
- **Complex dependence** between individual securities, modelled via copula

Motivation: how to estimate quantile

- **Check function**: Koenker & Bassett (1978, Econometrica)

$$\rho_p(x) = |x| \{pI_{[x\geq 0]} + (1-p)I_{[x<0]}\}, \quad 0 < p < 1$$

$$\hat{q}_p(X) = \arg\min_\theta \sum_{i=1}^n \rho_p(X_i - \theta)$$



- **likelihood function:**

  p.d.f.: asymmetric Laplace distribution

$$f_p(x) = p(1-p)\exp\{-\rho_p(x)\}$$

$$\hat{q}_p(X) = \arg\max_\theta \prod_{i=1}^n f_p(X_i - \theta)$$

c.f., Yu, Lu & Stander (2003, JRSS, series D)

Quantile regression: applications and current research areas

Asymmetric Laplace Distribution

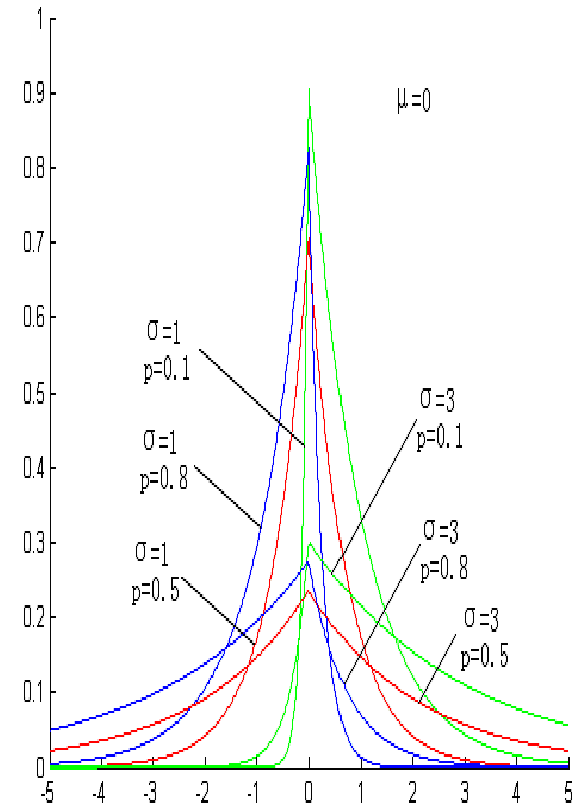
- ALD: density function

$$f(x\,|\,\sigma,p)=\frac{k}{\sigma}\exp\left\{-\left(\frac{1}{1-p}I_{[x>0]}+\frac{1}{p}I_{[x<0]}\right)\frac{k}{\sigma}\,|x|\right\}$$

where $k=k(p)=\sqrt{p^2+(1-p)^2}$

- σ -standard deviation， p -shape parameter in (0,1).

- Laplace distribution: p=0.5.



Skewed EWMA and forecasting

- **Skewed** EWMA Volatility forecasting

Lu & Huang (2007):

$$VaR_{t+1} = \frac{p_{t+1}}{k_{t+1}} \sigma_{t+1} \ln \frac{1-\alpha}{p_{t+1}} \quad , \quad k_{t+1} = \sqrt{{p_{t+1}}^2 + (1-p_{t+1})^2}$$

$$\sigma_{t+1} = \lambda \sigma_t + (1-\lambda)\left(\frac{k}{1-p} I_{[r_t>0]} + \frac{k}{p} I_{[r_t<0]} \right) | r_t | \quad \Leftarrow \hat{\sigma} = \frac{k}{T} \sum_{i=1}^{T} \left(\frac{1}{p} I_{[r_i<0]} + \frac{1}{1-p} I_{[r_i>0]} \right) | r_i |$$

$$p_{t+1} = \frac{1}{1 + \sqrt{u_{t+1} / v_{t+1}}},$$

$$u_{t+1} = \beta u_t + (1-\beta) | r_t | I_{[r_t<0]} \quad \Leftarrow u = \frac{1}{n} \sum_{i=1}^{n} | r_i | I_{[r_i>0]}$$

$$v_{t+1} = \beta v_t + (1-\beta) | r_t | I_{[r_t>0]} \quad \Leftarrow v = \frac{1}{n} \sum_{i=1}^{n} | r_i | I_{[r_i<0]}$$

Joint distribution via copula

❑ Archimedean copulas

- **Definition :** Let $\phi:[0,1]\to[0,\infty]$ be a continuous, strictly decreasing, convex function such that $\phi(1)=0$ and let $\phi^{[-1]}(t)$ be its pseudo inverse. Then

$$C(u,v) = \phi^{[-1]}(\phi(u)+\phi(v))$$

is an Archimedean copula.

- Clayton copula function fits the dataset best

$$\phi(t) = t^{-\theta} - 1 \qquad\qquad C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}$$

Evaluation of VaR forecasting

7 financial return series: **1 Jan. 1992 to 31 Dec. 2001**

Name	mean	s.d.	Skew	kurtosis	sample-size
BU	0.0099	0.5557	0.3101	5.8632	2533
CU	0.0127	0.3138	−0.1273	5.1645	2533
JU	0.0019	0.7381	−0.5929	7.5661	2533
GO	0.0095	0.7765	−1.4176	28.4840	2519
HK	0.0394	1.8531	0.0180	11.0592	2474
SH	0.0700	3.2021	5.4998	114.7436	2468
SZ	0.0504	2.4560	0.6617	13.2033	2458

- **Confidence level for VaR:** α =99%
- ***Likelihood criterion***: First 500 observations (estimation sample)
  β =0.998， λ taking 15 values between 0.85 and 0.99.

Backtesting methods:

❖ Kupiec(1995) and Christofferson (1998) likelihood ratio test:

Let $I_t = \begin{cases} 1, if\ r_t < VaR_t \\ 0, otherwise \end{cases}$, $N = \sum_{t=1}^{T} I_t$, T – sample size

➢ ***Unconditional coverage test***: $\tau = 1 - \alpha$

Assume $\{I_t\}$ independent,

H_0: $E(I_t) = \tau$ H_1: $E(I_t) \neq \tau$

$$LR_u = -2\ln[(1-\tau)^{T-N}\tau^N] + 2\ln[(1-N/T)^{T-N}(N/T)^T] \sim \chi^2(1)$$

- *Independence* test:

  H_0: $\{I_t\}$ independent

  H_1: $\{I_t\}$is a first order Markov process

$$LR_{in} = -2\ln[(1-\pi)^{T_{00}+T_{10}}\pi^{T_{01}+T_{11}}] + 2\ln[\pi_{00}^{T_{00}}\pi_{01}^{T_{01}}\pi_{10}^{T_{10}}\pi_{11}^{T_{11}}] \sim \chi^2(1)$$

- *Conditional coverage* test:

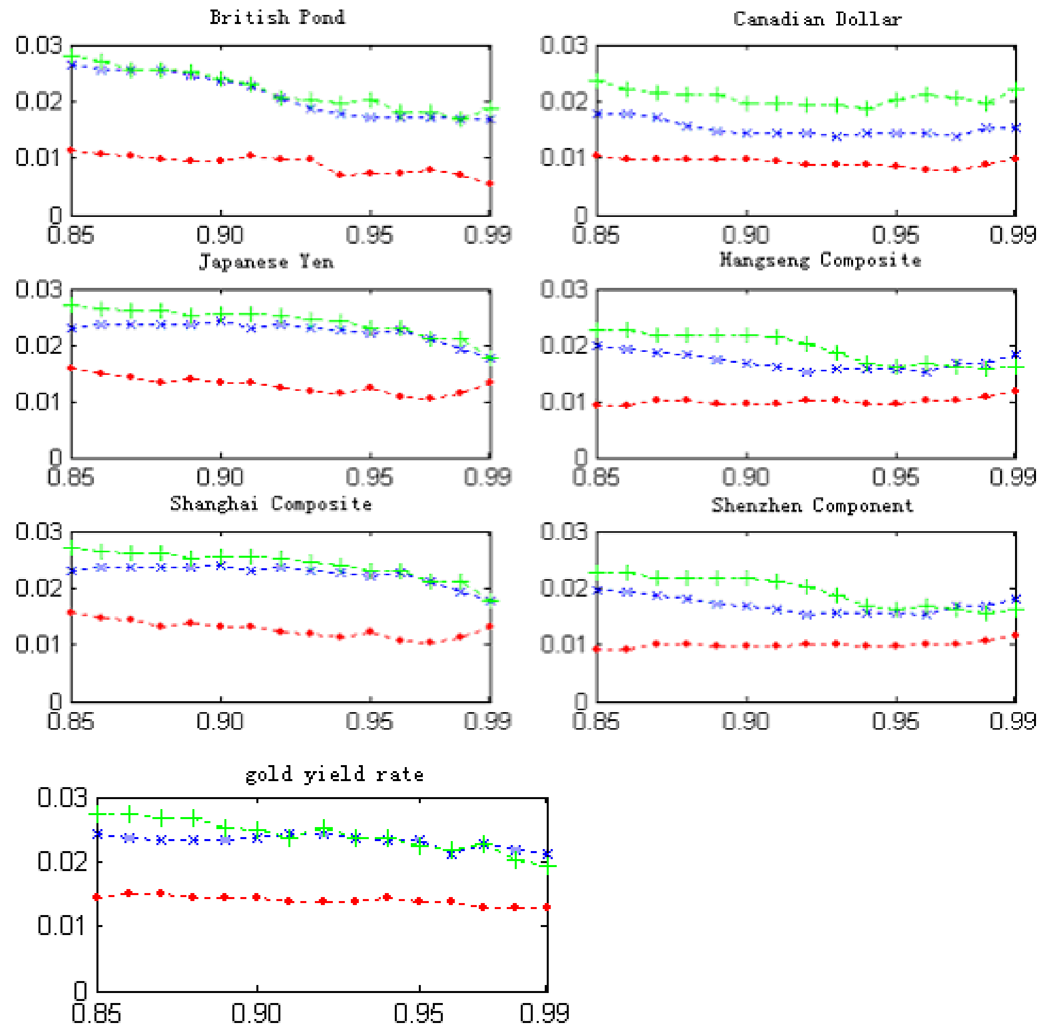
  H_0: $E(I_t) = \tau$

  H_1: $\{I_t\}$ a first order Markov process

$$LR_c = LR_u + LR_{in} \sim \chi^2(2)$$

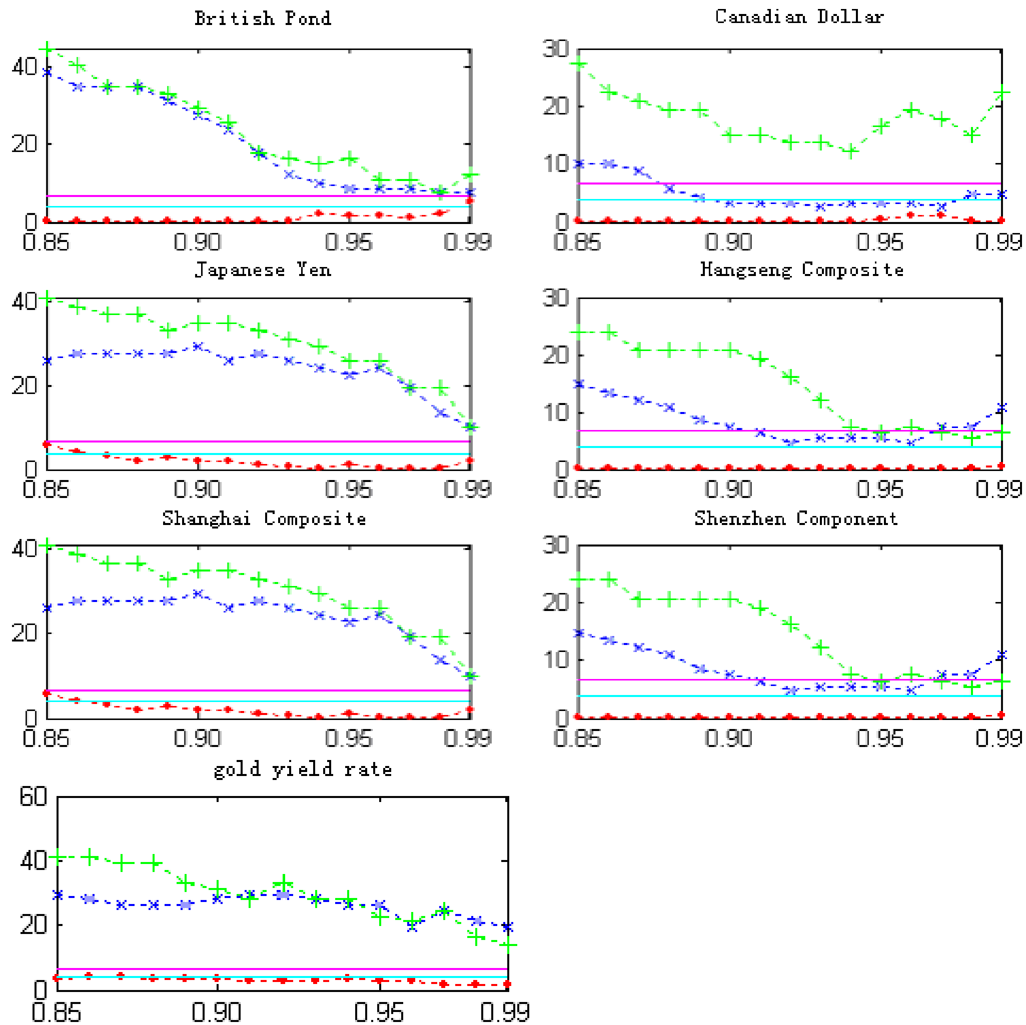
Unconditional Coverage

- Y axis-*failure rate* of VaR (theoretical 1%)
- X axis- value of λ
- St-EWMA +;
  R-EWMA x;
  Sk-EWMA •
- Sk-EWMA: about 1% ;
  St- and R-EWMA: >>1%



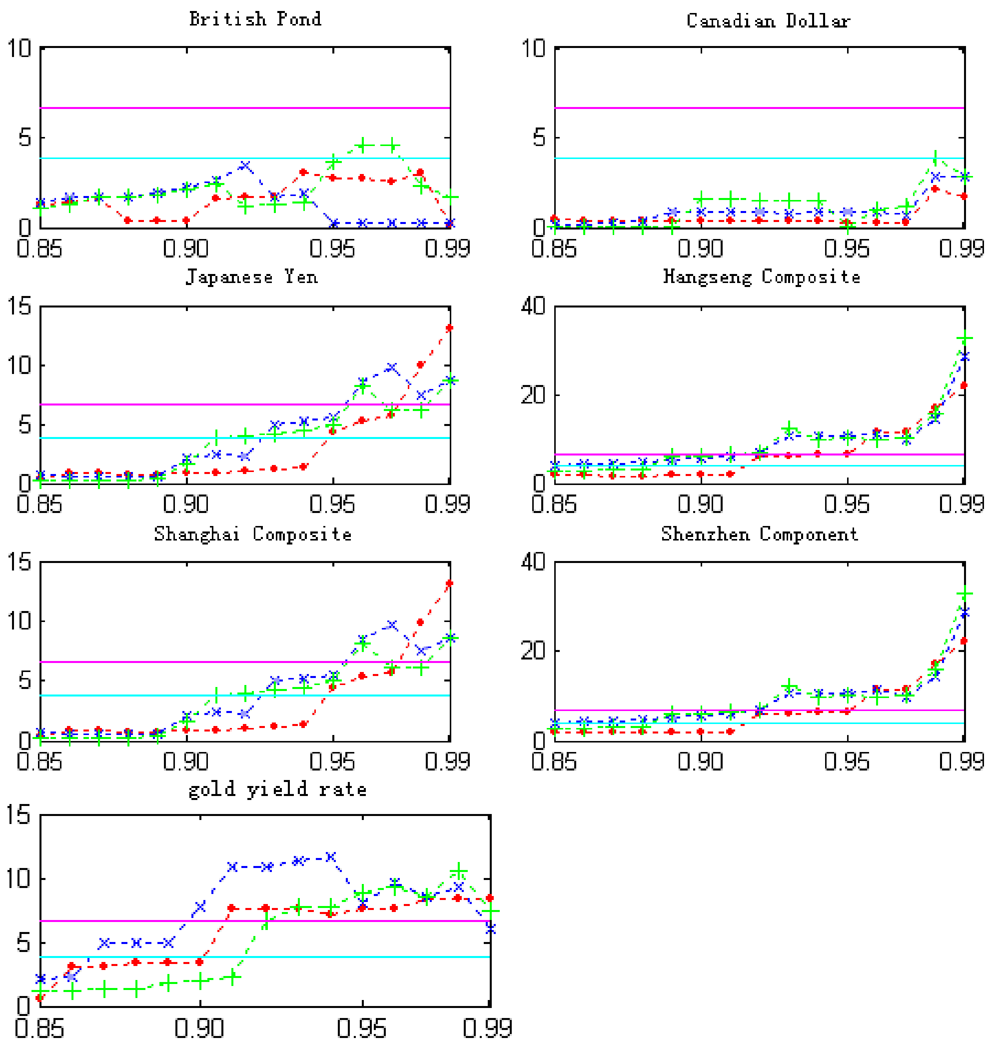
Unconditional Coverage test

- Y axis-*Likelihood ratio value*
- X axis- value of λ
- St-EWMA +;
  R-EWMA x;
  Sk-EWMA •
- Sk-EWMA: all less than 1% critical value of 6.63 (pink);
  St- and R-EWMA: mostly larger than 1% critical value (pink).



Independence test

- Y axis- *Likelihood ratio value*
- X axis- value of λ
- St-EWMA +;
  R-EWMA x;
  Sk-EWMA •
- 5% critical value 3.84 (blue) ;
  1% critical value 6.63 (pink)



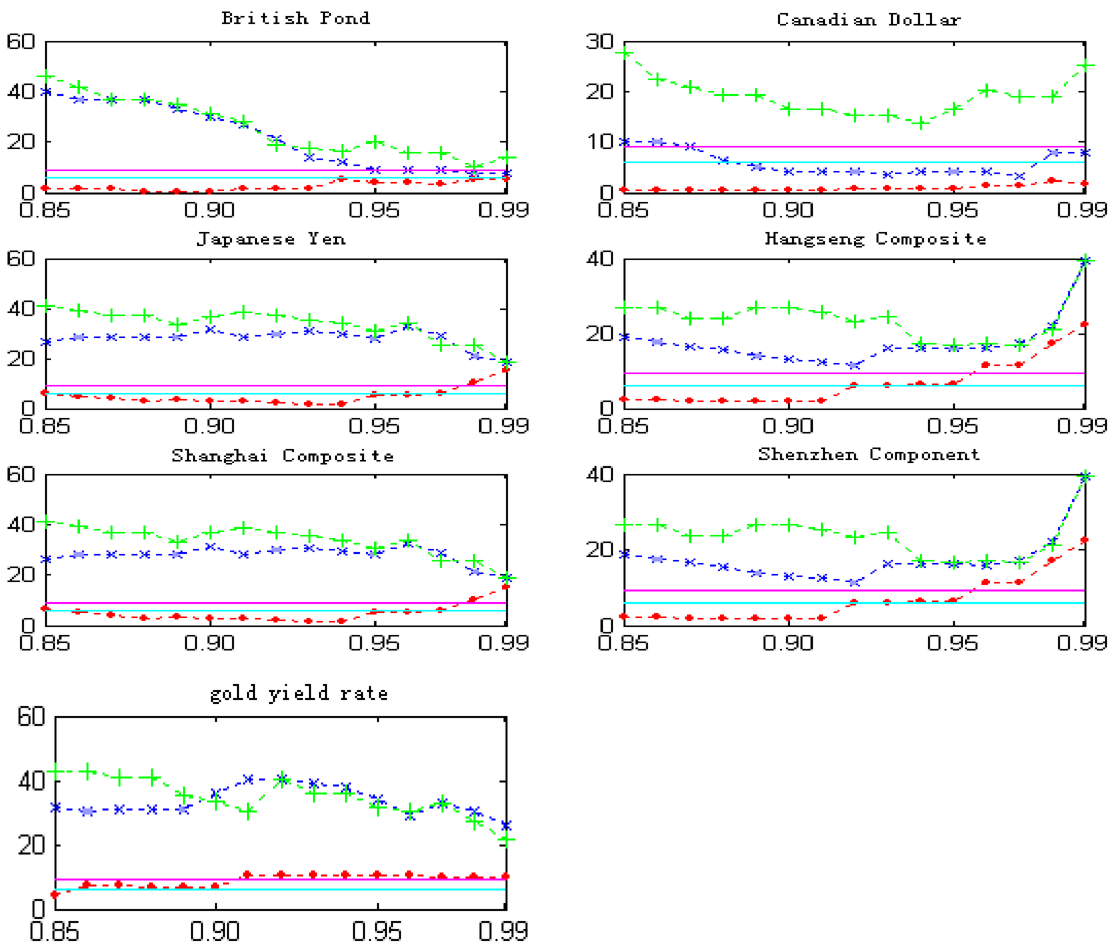
Conditional coverage test

- Y axis-Likelihood ratio value
- X axis- value of λ
- St-EWMA +;

  R-EWMA x;

  Sk-EWMA •
- Sk-EWMA: less than 1% critical value of 9.21 (pink)

  St- and R-EWMA: mostly larger than 1% critical value (pink)



Portfolio via copula

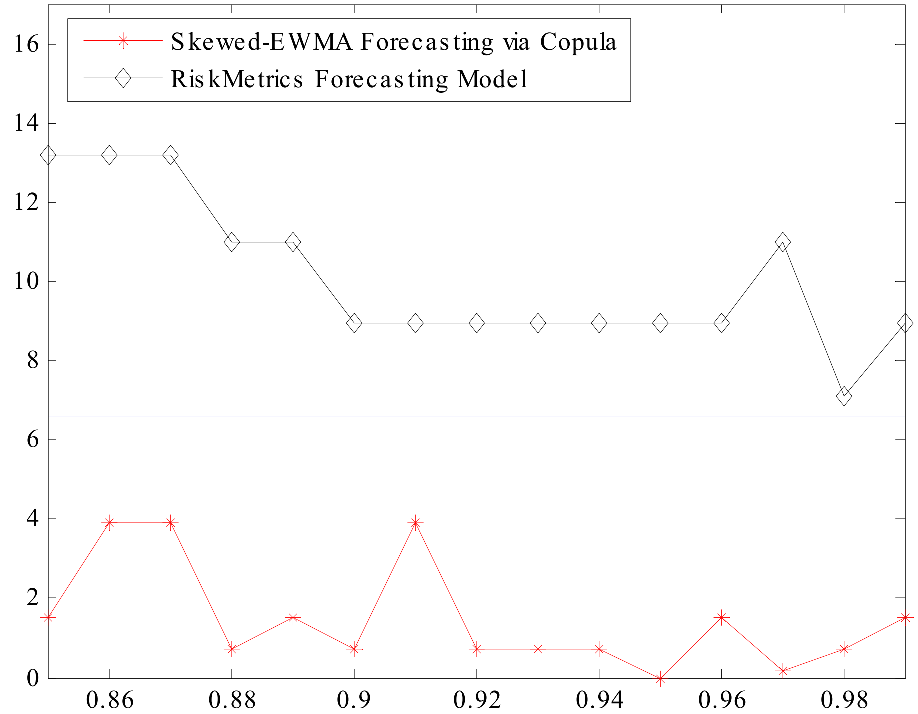
- Portfolio: Shanghai composite index and gold yield (half-half)
- Empirical test: Clayton copula function fits the dataset best

$$C(u,v)=\left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta}$$

Portfolio via copula:

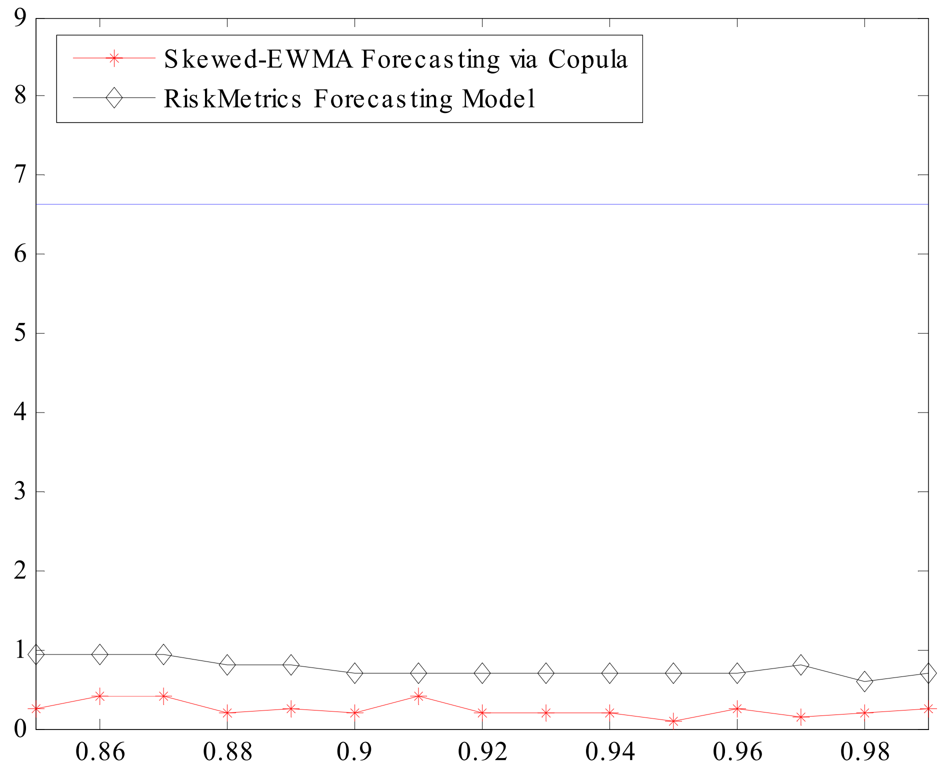
Unconditional Coverage test

- Y axis-*Likelihood ratio value*
- X axis- value of λ
- Sk-EWMA via copula: LR is less than 1% critical value of 6.63 ;
- RiskMetrics: LR is larger than 1% critical value.



Portfolio via copula:
Independence test

- Y axis- *Likelihood ratio value*
- X axis- value of λ
- 1% critical value 6.63 (blue)

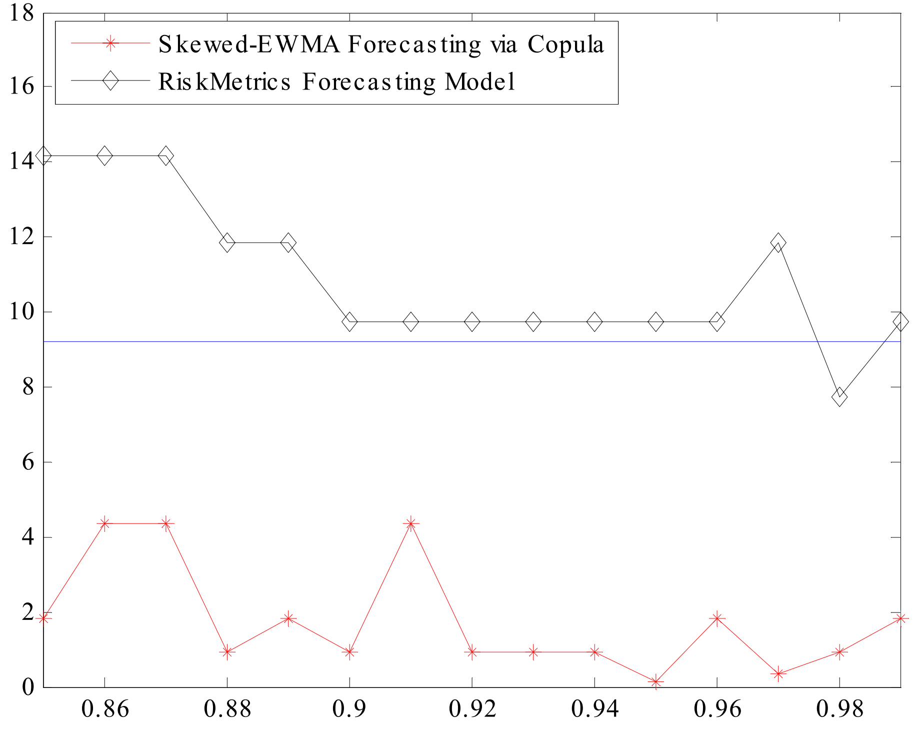


Portfolio via copula:

Conditional coverage test

- Y axis-Likelihood ratio value
- X axis- value of λ
- Sk-EWMA: LR is less than 1% critical value of 9.21(pink);

RiskMetrics: LR is larger than 1% critical value.



IV. Further work and outlook

- Combining skewed-EWMA with historical simulation, Monte Carol simulation and extreme value theory
  - Hull,J. and White, A., 1998b, "Incorporating volatility updating into the historical simulation method for Value-at-Risk", Journal of Risk, 1, 5-19.
- Extending to risk modeling beyond market risk:
  - E.g., credit risk, insurance risks