SYNOPSIS

STOCHASTIC ORDERING OF REINSURANCE STRUCTURES
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Key words: Reinsurance; Usual Stochastic Order, Convex Order; Increasing Convex Order; Stochastic Dominance; Insurance Premium Principles

Purpose of your paper: The paper offers a simple framework for ranking the common reinsurance structures in practice with the theory of stochastic orders. The basic idea is to slice the space of reinsurance structures into groups by expected loss cost to facilitate the comparisons within the group and between groups. Given the standard risk aversion assumption in economics, a spectrum of reinsurance structures with the same expected loss cost can be compared analytically with one another and sequenced based on their risk coverage under the convex order. The paper then expands the dimension of the comparison to groups of reinsurance structures with different expected loss costs, which can be ranked under the increasing convex order and the usual stochastic order. As such, the paper maps out the ordering for the entire space of reinsurance structures and presents it in a matrix format for quick reference.

Synopsis: Reinsurance is one of the most frequently used risk management tools by insurance companies in managing their portfolios. Insurance companies regularly evaluate and if necessary, modify the structure of their reinsurance program to adjust their overall risk exposures in an evolving business environment. For example, an enterprise risk management (ERM) analysis may compare the coverage and the efficiency between the current reinsurance program and alternative reinsurance structures. These alternative reinsurance structures may involve increasing or decreasing the retention level of an excess of loss reinsurance, adding an aggregate deductible or an aggregate limit, and adjusting the placement ratio.

To find the optimal reinsurance that maximizes an objective variable, such as the net underwriting income, the typical industry approach is to run a simulation model with as many potential reinsurance structures as possible. One of the key challenges in the ERM evaluation process is how to set the reinsurance prices for these alternative options, which to a large extent determines the efficiencies of the options. Given that the ERM modelers usually do not have the benefit of market quotes for all the options, it is important that these reinsurance structures can be properly ordered and priced in the model. The abundance of reinsurance choices together with the complexity of reinsurance pricing, however, often makes the selection process very difficult.

The goal of this paper is to provide actuaries, underwriters, and brokers a framework to compare common reinsurance structures so that unnecessary simulation may be avoided and reasonable results can be obtained quickly in an ERM analysis. We first explore the risk ranking of common reinsurance structures using the convex order from the theory of stochastic orders (e.g., Shaked and Shanthikumar (2007), Muller and Stoyan (2002), and Denuit et al. (2005)). We then further expand the dimension of the comparison to reinsurance structures with different expected loss costs using the usual stochastic order (equivalently, the First-order Stochastic Dominance) and the increasing convex order (dual to the general Second-order Stochastic Dominance) (See Levy (1998) for a general introduction to stochastic dominance, and see Heyer (2001) for an application of stochastic dominance to reinsurance.).

The convex order is dual to the concave order, which is the familiar Rothschild-Stiglitz second-order stochastic dominance (R-S SSD) with equal means as pioneered by Rothschild and Stiglitz (1970) in economics. Heyer (2001) uses the general SSD to rank reinsurance contracts on an empirical distribution basis through simulation. Assuming a risk-averse principal (or equivalently an increasing
concave utility function), if the net underwriting income resulting from reinsurance structure A is larger in "size" and less volatile than reinsurance structure B, then A is second-order stochastic dominating B from a cedant's point of view. However, the result of the underwriting income comparison using the general SSD is often inconclusive as demonstrated in Heyer's analysis. This paper will focus on the loss distributions, rather than the underwriting income distributions, of the reinsurance structures as there exists a natural ordering for the former, but not necessarily for the latter.

The convex order allows us to compare alternatives that have the same expected value, and thus eliminate the need to compare "size" or "magnitude." The focus of the comparison, instead, can then be on the "variability" or the pure risk of the reinsurance structures. We will show analytically that any risk-averse individual under the convex order can distinguish and rank basic reinsurance structures given their natural orders in "variability." In short, under the convex order, the stop-loss reinsurance is more risky than the quota share reinsurance, which in turn is more risky than the reinsurance with an aggregate limit (i.e., 100\% quota share with a cap):

\[
\text{Aggregate Limit} \leq_{cx} \text{Quota Share} \leq_{cx} \text{Stop-loss}
\]

where \( A \leq_{cx} B \) means B dominates A under the convex order.

This line of reasoning can be extended to analyzing the aggregate loss treaties with more than one contract feature. For example, a quota share treaty with a stop-loss threshold can be compared with a quota share treaty with an aggregate limit. More parameters need to be calibrated within a treaty to make sure that the mean loss is the same across all treaties as required by the convex order. Note that these combination structures with two contract features form a continuum of options that are bounded by the three basic reinsurance structures. Outlined below are the rankings of some possible combinations.

\[
\begin{align*}
\text{Aggregate Limit} & \leq_{cx} \text{Mixture of Quota Share and Aggregate Limit} \\
& \leq_{cx} \text{Quota Share} \\
& \leq_{cx} \text{Mixture of Stop-loss and Quota Share} \\
& \leq_{cx} \text{Stop-loss.}
\end{align*}
\]

The approaches we have used in analyzing the aggregate loss reinsurance can also be applied to the excess of loss (XOL) reinsurance treaties with features such as annual aggregate deductible (AAD), higher per claim retention, partial placement (or equivalently cedant co-participation) and aggregate limit. Note that the convex order is closed under convolutions. That is, when the claim count distribution is independent of the severity distributions, the dominance relationship between the severity distributions at the per risk/per occurrence level can be carried over to the aggregate layer loss level. The closure property is important in proving the relationship between XOL with Partial Placement and XOL with Higher Retention. We will show that under the convex order, seven types of XOL reinsurance treaties along with the corresponding hybrid structures can be ranked analytically.

The next step is to expand the dimension of the comparison to reinsurance structures with different expected loss costs using the usual stochastic order (equivalently, the First-order Stochastic Dominance) and the increasing convex order (the dual to the general Second-order Stochastic Dominance). The usual stochastic order \( (\leq_{st}) \) can be established between any two structures that are of the same type, but have different expected losses. If different types of structures are involved in the one-on-one comparison, we may be able to establish dominance under the weaker increasing convex order \( (\leq_{icx}) \).
The use of the usual stochastic order and increasing convex order greatly expands the range of reinsurance structures that can be compared and ranked. While it appears that the number of comparison combinations may be infinite, some reinsurance treaties, however, are not comparable under any of the three stochastic orders. Particularly, the comparison is inconclusive between a quota share treaty and a treaty with both an aggregate limit and an aggregate deductible. The reason for inconclusiveness is that neither treaty has thicker tails on both ends of the density function, which is required for the dominance relationship. But we will show that the inconclusiveness follows a predictable pattern based on the types of reinsurance structure. In the appendix, we analyze some XOL reinsurance structures that cannot be compared with those structures analyzed in Section 5. The implication is that we may need to divide reinsurance structures into subsets such that the members of the subsets can be compared with one another.

References


