Managing Systematic Mortality Risk in Life Annuities: An Application of Longevity Derivatives

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Introduction

Provision of Retirement Income Products

- Examples of retirement income products: life annuities, deferred annuities, variable annuities with guarantees (financial options)
- Selling them can be risky - long term nature
- Duration of payments depend on survival of annuitants
- Idiosyncratic mortality risk can be handled by diversification
- Systematic mortality risk (longevity risk) is significant
- Hedging longevity risk using longevity swaps and caps
"Pricing and hedging analysis of longevity derivatives on a hypothetical life annuity portfolio subject to longevity risk"

- Propose and calibrate a two-factor Gaussian mortality model
- Derive analytical pricing formulas for longevity swaps and caps
- Investigate hedging features of longevity swaps and caps w.r.t. different assumptions on (1) the market price of longevity risk (2) term to maturity of the hedging instruments (3) portfolio size
Longevity risk modelling

How to model survival probability? Intensity-based approach:

- Cox process with stochastic intensity $\mu$. If $\mu$ is known then deal with inhomogeneous Poisson process, where probability of $k$ events in $[t, T]$ is given by

$$
\mathbb{P}(N_T - N_t = k \mid \mathcal{F}_T) = \frac{\left( \int_t^T \mu(s) \, ds \right)^k}{k!} e^{-\int_t^T \mu(s) \, ds} \quad (1)
$$

- Death time is modelled as the first jump time of a Cox process

- Given only $\mathcal{F}_t$, we set $k = 0$ in Eq. (1) and use iterated expectation to obtain the expected survival probability

$$
S_{x+t}(t, T) = E_{\mathbb{P}} \left( e^{-\int_t^T \mu_{x+s}(s) \, ds} \mid \mathcal{F}_t \right) \quad (2)
$$

Useful for pricing and the corresponding density function of death time can be obtained
Mortality model

The mortality intensity $\mu_x(t)$ (in full: $\mu_{x+t}(t)$) of a cohort aged $x$ at time $t = 0$ is modelled by

$$\mu_x(t) = Y_1(t) + Y_2(t),$$

where $Y_1(t)$ is a general trend that is common to all ages, and $Y_2(t)$ is an age-specific factor, satisfying the following dynamics

$$dY_1(t) = \alpha_1 Y_1(t) \, dt + \sigma_1 \, dW_1(t)$$

$$dY_2(t) = (\alpha x + \beta) Y_2(t) \, dt + \sigma e^{\gamma x} \, dW_2(t)$$

where $dW_1(t) \, dW_2(t) = \rho \, dt$
Survival probability

Proposition

Under the two-factor mortality model, the expected \((T-t)\)-survival probability of a person aged \(x + t\) at time \(t\) is given by

\[
S_{x+t}(t, T) = E_t^P \left( e^{-\int_t^T \mu_x(s) ds} \right) = e^{\frac{1}{2} \Gamma(t, T) - \Theta(t, T)}
\]

(6)

where \(\Theta(t, T)\) and \(\Gamma(t, T)\) are the mean and the variance of the integral \(\int_t^T \mu_x(s) ds\) respectively.

We will use the fact that the integral \(\int_t^T \mu_x(s) ds\) is normally distributed with known mean and variance to derive analytical pricing formulas for longevity derivatives.
Parameter estimation

**Figure:** Central death rates $m(x, t)$.

- **Australian male ages** $x = 60, ..., 95$ and years $t = 1970, ...2008$
- **Intensity assumed constant over each integer age and calendar year; approximated by central death rates** $m(x, t)$
Parameter estimation

Figure: Difference of central death rates $m(x, t)$.

- Sample variance: $\text{Var}(\Delta m_x)$; Model variance: $\text{Var}(\Delta \mu_x) = (\sigma_1^2 + 2\sigma_1 \sigma \rho e^{\gamma x} + \sigma^2 e^{2\gamma x}) \Delta t$
- Minimizing $\sum_{x=60, 65 \ldots}^{90} (\text{Var}(\Delta \mu_x | \sigma_1, \sigma, \gamma, \rho) - \text{Var}(\Delta m_x))^2$ with respect to the parameters $\{\sigma_1, \sigma, \gamma, \rho\}$. 
Other parameters are calibrated to the empirical survival curves aged 65 and 75 in 2008 by minimizing

$$
\sum_{x=65,75} \sum_{j=1}^{T_x} \left( \hat{S}_x(0,j) - S_x(0,j) \right)^2
$$

(7)
Risk-adjusted measure

- Assuming, under the risk-adjusted measure $\mathbb{Q}$, we have

\[
dY_1(t) = \alpha_1 Y_1(t) \, dt + \sigma_1 \, d\tilde{W}_1(t)
\]

\[
dY_2(t) = (\alpha x + \beta - \lambda \sigma e^{\gamma x}) Y_2(t) \, dt + \sigma e^{\gamma x} \, d\tilde{W}_2(t)
\]

where $\lambda$ is the market price of longevity risk; $\lambda$ is estimated using the proposed price of the BNP/EIB longevity bond (Meyricke & Sherris (2014))
(Index-based) Longevity swaps

- Consider an annuity provider/hedger who has a future liability that depends on the (stochastic) survival probability of a cohort
- The provider wants certainty for the estimated survival probability
- S-forward: At maturity $T$, pays fixed leg, $K(T) \in (0, 1)$, and receives floating leg - the realized survival probability $e^{-\int_0^T \mu_x(s) \, ds}$.
- The payoff from the S-forward is, assuming the notional amount is 1,

$$e^{-\int_0^T \mu_x(s) \, ds} - K(T)$$  \hspace{1cm} (10)

- Zero price at inception means that

$$e^{-rT} E_0^Q \left( e^{-\int_0^T \mu_x(s) \, ds} - K(T) \right) = 0$$  \hspace{1cm} (11)

and hence

$$K(T) = E_0^Q \left( e^{-\int_0^T \mu_x(s) \, ds} \right)$$  \hspace{1cm} (12)

- A longevity swap is a portfolio of S-forwards
The mark-to-market price process $F(t)$ of an $S$-forward is

$$
F(t) = e^{-r(T-t)} E_t^Q \left( e^{-\int_0^T \mu_x(s) \, ds} - K \right)
$$

$$
= e^{-r(T-t)} E_t^Q \left( e^{-\int_0^t \mu_x(s) \, ds} e^{-\int_t^T \mu_x(s) \, ds} - K \right)
$$

$$
= e^{-r(T-t)} (\bar{S}_x(0, t) \tilde{S}_{x+t}(t, T) - K) \tag{13}
$$

$\bar{S}_x(0, t) := e^{-\int_0^t \mu_x(s) \, ds} |_{\mathcal{F}_t}$ is the realized survival probability, which is observable given $\mathcal{F}_t$. Let $\hat{n}$ denotes the number of survivors in $[0, t]$ and the initial population of the cohort is $n$, then we have

$$
\bar{S}_x(0, t) \approx \frac{\hat{n}}{n} \tag{14}
$$

$\tilde{S}_{x+t}(t, T)$ is the risk-adjusted survival probability; If an analytical expression for $\tilde{S}_{x+t}(t, T)$ is available then an explicit formula for $F(t)$ is obtained
(Index-based) Longevity caps

A longevity cap is a portfolio of caplets - the payoff of a longevity caplet is

\[
\max \left\{ \left( e^{-\int_0^T \mu_x(s) \, ds} - K \right), 0 \right\}
\]

(15)

Similar to an S-forward but is able to capture the upside potential - when survival probability is overestimated

The price of the longevity caplet:

\[
C_{\ell}(t) = e^{-r(T-t)} E_t^Q \left( \left( e^{-\int_0^T \mu_x(s) \, ds} - K \right) \right)^+
\]

Proposition

Under the two-factor mortality model, the price at time \( t \) of a longevity caplet \( C_{\ell}(t) \), with maturity \( T \) and strike \( K \), is given by

\[
C_{\ell}(t) = \bar{S}_t \tilde{S}_t e^{-r(T-t)} \Phi \left( \sqrt{\tilde{\Gamma}(t, T)} - d \right) - Ke^{-r(T-t)} \Phi (-d)
\]

(16)

where \( \bar{S}_t = \bar{S}_x(0, t), \tilde{S}_t = \tilde{S}_{x+t}(t, T) \), \( d = \frac{1}{\sqrt{\tilde{\Gamma}(t, T)}} \left( \ln \left\{ K / (\bar{S}_t \tilde{S}_t) \right\} + \frac{1}{2} \tilde{\Gamma}(t, T) \right) \)

and \( \Phi(\cdot) \) denotes the CDF of the standard normal R.V.
Idea of proof: since the integral $I := \int_t^T \mu_x(s) \, ds$ is normally distributed, $e^{-I}$ is a log-normal random variable.

The standard deviation $\sqrt{\tilde{\Gamma}(t, T)}$ of the integral can be interpreted as the volatility of the (risk-adjusted) aggregated longevity risk for an individual aged $x + t$ at time $t$, for the period from $t$ to $T$.

**Figure:** Caplet price as a function of (left panel) $T$ and $K$ and (right panel) $\lambda$ where $K = 0.4$ and $T = 20$. 
Managing longevity risk in a life annuity portfolio

- Consider a life annuity portfolio consists of \( n \) policyholders aged 65 in year 2008; Premium of the annuity ($1 per year upon survival) is given by

\[
a_x = \sum_{T=1}^{\omega-x} B(0, T) \tilde{S}_x(0, T; \lambda)
\]  

(17)

- For the annuity provider the present value (P.V.) of asset is \( A = n a_x \)

- P.V. of (random) liability for policyholder \( k \):

\[
L_k = \sum_{T=1}^{\lfloor \tau_k \rfloor} B(0, T)
\]  

(18)

- P.V. of liability for the whole portfolio: \( L = \sum_{k=1}^{n} L_k \)
Interested in the discounted surplus distribution per policy

\[ \frac{D_{no}}{n} \]  

where \( D_{no} = A - L \)

**Figure:** No longevity risk \((\sigma_1 = \sigma = 0)\); Idiosyncratic mortality risk is reduced as \( n \) increases
**Swap-hedged annuity portfolio:**

\[ D_{\text{swap}} = A - L + F_{\text{swap}} \]  

(20)

where

\[ F_{\text{swap}} = n \sum_{T=1}^{\hat{T}} B(0, T) \left( e^{-\int_0^T \mu_x(s) \, ds} - \tilde{S}_x(0, T) \right) \]  

(21)

is the random cash flow from the longevity swap; \( n \) here acts as the notional amount; \( \hat{T} \): term to maturity

**Cap-hedged annuity portfolio:**

\[ D_{\text{cap}} = A - L + F_{\text{cap}} - C_{\text{cap}} \]  

(22)

where

\[ F_{\text{cap}} = n \sum_{T=1}^{\hat{T}} B(0, T) \max \left\{ \left( e^{-\int_0^T \mu_x(s) \, ds} - S_x(0, T) \right), 0 \right\} \]  

(23)

is the random cash flow from the longevity cap and

\[ C_{\text{cap}} = n \sum_{T=1}^{\hat{T}} C_{\ell}(0; T, S_x(0, T)) \]  

(24)

is the price of the longevity cap
Examining hedge features of longevity swaps and caps with respect to (w.r.t.) different assumptions on (1) the market price of longevity risk $\lambda$ (2) term to maturity of the hedging instruments $\hat{T}$ and (3) the portfolio size $n$

**Table:** Parameters for the base case.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\hat{T}$ (years)</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.5</td>
<td>30</td>
<td>4,000</td>
</tr>
</tbody>
</table>
**Figure**: Effect of the market price of longevity risk $\lambda$ on the discounted surplus distribution per policy.
<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std.dev.</th>
<th>Skewness</th>
<th>VaR$_{0.99}$</th>
<th>ES$_{0.99}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>$\lambda = 0$</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No hedge</td>
<td>-0.0076</td>
<td>0.3592</td>
<td>-0.2804</td>
<td>-0.9202</td>
<td>-1.1027</td>
</tr>
<tr>
<td>Swap-hedged</td>
<td>-0.0089</td>
<td>0.0718</td>
<td>-0.1919</td>
<td>-0.1840</td>
<td>-0.2231</td>
</tr>
<tr>
<td>Cap-hedged</td>
<td>-0.0086</td>
<td>0.2054</td>
<td>1.0855</td>
<td>-0.3193</td>
<td>-0.3515</td>
</tr>
<tr>
<td><strong>$\lambda = 4.5$</strong></td>
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<td></td>
</tr>
<tr>
<td>No hedge</td>
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<td>0.3592</td>
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<td>-0.2094</td>
</tr>
<tr>
<td>Cap-hedged</td>
<td>0.0682</td>
<td>0.2054</td>
<td>1.0855</td>
<td>-0.2425</td>
<td>-0.2746</td>
</tr>
<tr>
<td><strong>$\lambda = 8.5$</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>No hedge</td>
<td>0.2978</td>
<td>0.3592</td>
<td>-0.2804</td>
<td>-0.6148</td>
<td>-0.7973</td>
</tr>
<tr>
<td>Swap-hedged</td>
<td>0.0204</td>
<td>0.0718</td>
<td>-0.1919</td>
<td>-0.1547</td>
<td>-0.1938</td>
</tr>
<tr>
<td>Cap-hedged</td>
<td>0.1205</td>
<td>0.2054</td>
<td>1.0855</td>
<td>-0.1903</td>
<td>-0.2224</td>
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<tr>
<td><strong>$\lambda = 12.5$</strong></td>
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<tr>
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<td>-0.4650</td>
<td>-0.6476</td>
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<tr>
<td>Swap-hedged</td>
<td>0.0398</td>
<td>0.0718</td>
<td>-0.1919</td>
<td>-0.1354</td>
<td>-0.1744</td>
</tr>
<tr>
<td>Cap-hedged</td>
<td>0.1619</td>
<td>0.2054</td>
<td>1.0855</td>
<td>-0.1489</td>
<td>-0.1810</td>
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</table>
**Figure:** Effect of the term to maturity $\hat{T}$ of the hedging instruments on the discounted surplus distribution per policy.
Table: Hedging features of a longevity swap and cap w.r.t. term to maturity $\hat{T}$.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std.dev.</th>
<th>Skewness</th>
<th>VaR&lt;sub&gt;0.99&lt;/sub&gt;</th>
<th>ES&lt;sub&gt;0.99&lt;/sub&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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</tr>
<tr>
<td><strong>$\hat{T} = 10$ Years</strong></td>
<td></td>
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<tr>
<td>No hedge</td>
<td>0.2978</td>
<td>0.3592</td>
<td>-0.2804</td>
<td>-0.6148</td>
<td>-0.7973</td>
</tr>
<tr>
<td>Swap-hedged</td>
<td>0.2820</td>
<td>0.2911</td>
<td>-0.3871</td>
<td>-0.5707</td>
<td>-0.7490</td>
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<tr>
<td>Cap-hedged</td>
<td>0.2893</td>
<td>0.2989</td>
<td>-0.2661</td>
<td>-0.5801</td>
<td>-0.7592</td>
</tr>
<tr>
<td><strong>$\hat{T} = 20$ Years</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>No hedge</td>
<td>0.2978</td>
<td>0.3592</td>
<td>-0.2804</td>
<td>-0.6148</td>
<td>-0.7973</td>
</tr>
<tr>
<td>Swap-hedged</td>
<td>0.1740</td>
<td>0.1794</td>
<td>-0.7507</td>
<td>-0.3656</td>
<td>-0.5061</td>
</tr>
<tr>
<td>Cap-hedged</td>
<td>0.2234</td>
<td>0.2310</td>
<td>0.2006</td>
<td>-0.3870</td>
<td>-0.5259</td>
</tr>
<tr>
<td><strong>$\hat{T} = 30$ Years</strong></td>
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<td>-0.2804</td>
<td>-0.6148</td>
<td>-0.7973</td>
</tr>
<tr>
<td>Swap-hedged</td>
<td>0.0204</td>
<td>0.0718</td>
<td>-0.1919</td>
<td>-0.1547</td>
<td>-0.1938</td>
</tr>
<tr>
<td>Cap-hedged</td>
<td>0.1205</td>
<td>0.2054</td>
<td>1.0855</td>
<td>-0.1903</td>
<td>-0.2224</td>
</tr>
<tr>
<td><strong>$\hat{T} = 40$ Years</strong></td>
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<tr>
<td>No hedge</td>
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<td>0.3592</td>
<td>-0.2804</td>
<td>-0.6148</td>
<td>-0.7973</td>
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<tr>
<td>Swap-hedged</td>
<td>-0.0091</td>
<td>0.0668</td>
<td>0.0277</td>
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<td>0.1999</td>
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<td>-0.1909</td>
<td>-0.2131</td>
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</table>
Table: Longevity risk reduction $R = 1 - \frac{\text{Var}(\bar{D}^*)}{\text{Var}(\bar{D})}$ of a longevity swap and a cap w.r.t. different portfolio size ($n$).

<table>
<thead>
<tr>
<th>$n$</th>
<th>2,000</th>
<th>4,000</th>
<th>6,000</th>
<th>8,000</th>
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</thead>
<tbody>
<tr>
<td>$R_{\text{swap}}$</td>
<td>92.6%</td>
<td>96.0%</td>
<td>97.2%</td>
<td>97.7%</td>
</tr>
<tr>
<td>$R_{\text{cap}}$</td>
<td>64.9%</td>
<td>67.3%</td>
<td>68.0%</td>
<td>68.6%</td>
</tr>
</tbody>
</table>

- $\text{Var}(\bar{D}^*)$ variance of discounted surplus for hedged portfolio
- $\text{Var}(\bar{D})$ variance of discounted surplus for unhedged portfolio
**Table:** Hedging features of a longevity swap and a cap w.r.t. portfolio size \((n)\).

<table>
<thead>
<tr>
<th></th>
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<tr>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td><strong>(n = 2,000)</strong></td>
<td></td>
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<tr>
<td>No hedge</td>
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<td>0.3646</td>
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<td>-0.6360</td>
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<td>Swap-hedged</td>
<td>0.0200</td>
<td>0.0990</td>
<td>-0.1615</td>
<td>-0.2120</td>
<td>-0.2653</td>
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<tr>
<td>Cap-hedged</td>
<td>0.1200</td>
<td>0.2160</td>
<td>0.9220</td>
<td>-0.2432</td>
<td>-0.2944</td>
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<tr>
<td><strong>(n = 4,000)</strong></td>
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<tr>
<td>No hedge</td>
<td>0.2978</td>
<td>0.3592</td>
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<td>-0.6148</td>
<td>-0.7973</td>
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<td>0.2054</td>
<td>1.0855</td>
<td>-0.1903</td>
<td>-0.2224</td>
</tr>
<tr>
<td><strong>(n = 6,000)</strong></td>
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<td>No hedge</td>
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<td>-0.1660</td>
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<td>Cap-hedged</td>
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<td>1.1519</td>
<td>-0.1639</td>
<td>-0.2051</td>
</tr>
<tr>
<td><strong>(n = 8,000)</strong></td>
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<tr>
<td>No hedge</td>
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<td>Swap-hedged</td>
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<td>Cap-hedged</td>
<td>0.1209</td>
<td>0.1992</td>
<td>1.1616</td>
<td>-0.1598</td>
<td>-0.1991</td>
</tr>
</tbody>
</table>
Summary

- The proposed two-factor Gaussian mortality model is capable of modelling mortality intensities of different ages simultaneously.
- Longevity swaps and caps can be priced analytically under the model; Standard derivation of the integral $\int_t^T \mu_x(s)\,ds$ plays the role of volatility in longevity derivatives pricing.
- Swap-hedged annuity portfolio is insensitive to the market price of risk $\lambda$.
- Longevity swaps and caps are markedly different as hedging instruments when term to maturity $\hat{T}$ or portfolio size $n$ is large.
Thank you for your attention