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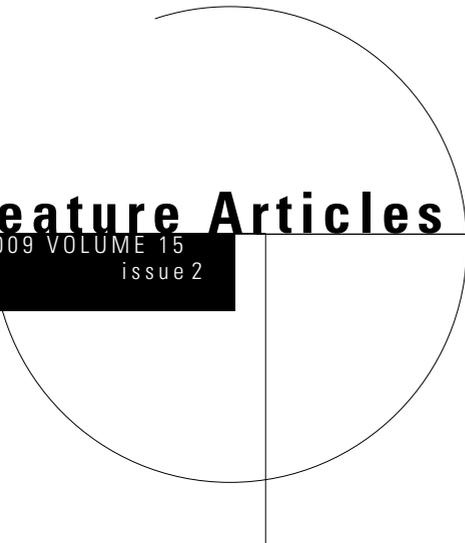


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Feature Articles

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Aboriginal and Torres Strait Islander Population: More than reported

*R C Madden & L R Jackson Pulver**

Abstract

A realistic estimate of population data and vital statistics such as births and deaths is essential to understanding the history and relative status of any population over time. Estimating the Aboriginal population of Australia has challenged statisticians for well over 100 years. Estimates of the 1788 population are reviewed, as well as estimates from colonial censuses and the early Yearbooks. The wide variation in contemporary estimates is discussed. More recent improvements in data and data quality are examined, and a plausible scenario for the size of the Aboriginal population over time, based on what we know today, is presented.

Keywords: Indigenous population, Aboriginal and Torres Strait Islander enumeration, Population statistics, Under enumeration, Backward projection ,

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1 Introduction

Estimation of Australia's Aboriginal and Torres Strait Islander population and characteristics has been, and remains, a difficult subject for statisticians. We argue that the population was always much more numerous than the official records report, until current times.

Estimates of the size of the Aboriginal population at various points in history are not simply of academic interest. Jackson Pulver (2003) has pointed out that it is important to acknowledge and chronicle 'historical events that have changed significantly the health of Aboriginal and Torres Strait Australians since 1788 and examine our shared history in order to find reasons for or rationale for the current circumstances of contemporary Aboriginal peoples' health.'

Knowing the size of a population is today seen as routine in placing historical events in context, but misstatement can completely distort public policy and decision making. Public policy over the years was made on the basis of the official estimates surveyed here. Much contemporary policy discussion reflected a view of a small and decreasing Aboriginal population around the time of Federation and afterwards (a 'rapidly disappearing race', 1911 Yearbook), and those policies have repercussions through to today. The 2001 Yearbook reports population 'Million Milestones', with the 1 million mark in 1858 and 3.8 million in 1901 (table 5.5). A footnote reports (incorrectly prior to 1901) that 'Aboriginal people were not included in population estimates prior to 1961.'

Undercounting has had significant repercussions for Aboriginal and Torres Strait Islander peoples. Discriminatory policies and inferior health and social conditions are less visible or worrying when they apply to a group that represents 2% of the population rather than 4%. Issues such as native title, removal of children and assimilation should have been debated against a sound estimation of the population size at various points in time. Continuing debate on these vital issues should now be informed by the most reliable population estimates over time.

We discuss the sources of the estimates of the Indigenous population over the years, and alternative views included in the official statistical publications of the time. We then examine the official reporting on Aboriginal population since World War II, and contemporary opinion about the population in 1788. We conclude with a plausible scenario for population numbers at various times based on current knowledge.

As a part of the historical survey we have undertaken, some terms are unavoidably used that are out of date and considered offensive to many. We apologise for any offence caused, it is not our intention.

2 Australian Yearbooks

Australian (formerly Commonwealth Bureau of Census and Statistics) have been published in almost all years since 1908. This series was preceded by a series of Yearbooks, starting in 1890, entitled the Seven Colonies of Australasia, prepared by Timothy Coghlan, the then NSW Statistician. These Coghlan Yearbooks included counts of the Aboriginal population obtained through censuses of the time. Several editions also included commentary on the census based counts, and gave alternate, higher estimates of the Aboriginal populations.

This paper examines the Yearbook reporting from 1890 onwards to draw out the comments over the more than 40 years from 1890 on the likely size of the Indigenous population and the comparison with census based estimates. Such commentary ceased after 1933. More recent official publications and estimates are presented. While a sophisticated backcasted population time series is beyond the scope of this paper, recent literature is examined to show the need for a systematic restatement of Indigenous population which is consistent with current estimates, soundly based partial evidence and the considered estimates of the early official statisticians.

3 Population Estimates at the time of European Settlement

The 1924 edition of the Australian Yearbook is the first to include an estimate of Aboriginal population in 1788, finding 150,000 people at that time. It attributes ‘a reasonable measure of certainty’ to this estimate, without further explanation or analysis.

A more thorough discussion is included in the 1930 Yearbook, which contains a special article by A.R. Radcliffe Brown, Professor of Anthropology at the University of Sydney. Brown goes through the data available for each State on the Aboriginal population at the time of white settlement, always adopting a conservative estimate. His estimate is a *minimum* of 251,000 people in 1788 (his emphasis). He concludes, stating:

The available evidence points to the original population of Australia having been certainly over 250,000, and quite possibly, or even probably, over 300,000.

The 1988 (Australia’s Bicentennial Year) Yearbook states that ‘recent archaeological finds suggest that a population of 750,000 could have been sustained.’ The 1994 Yearbook contains a special article on the Indigenous population which states

Various estimates, ranging from around 300,000 to over one million, have been made of the size of the Aboriginal and Torres Strait Islander population of Australia at the time of European settlement.

Subsequent yearbooks are no more explicit.

In a 1999 conference on Aboriginal population, Mulvaney (2002) comprehensively reviewed Radcliffe Brown’s estimate, as well as others that had appeared since 1930. Issues that he gives attention to include:

- increased recognition that the country before European settlement had had greater capacity to sustain people than earlier estimated; relevant factors included the role of female food gatherers in the provision of food supplies and the impact of firing land in increasing productivity of the land

- the extent to which European settlement had introduced factors that acted to decrease the population; these included the impact of diseases such as smallpox (which possibly emanated from Macassan fishermen visiting northern Australia (Campbell 2002)), influenza, TB and several other diseases after 1788 (these effects on communities could largely occur before permanent settlement affected them), increased recognition of the impact of frontier violence and decreased fertility.

Despite the obvious effect of the introduction of disease and the new dominion, Mulvaney mentions estimates as high as 1.25 million, but he concludes that 750,000-800,000 is the most reasonable range. Importantly, the range appears to be mis-printed in the reference, but Smith's following paper in the monograph reporting the conference proceedings contains the range quoted here (Smith 2002).

4 Contemporary Population estimates

4.1 1890 to 1945

Coghlan in his 1890, 1894 and 1896 Yearbooks estimated the Indigenous population of Australia to be 'something like 200,000', although the 1891 census enumerated only 38,879 Aboriginal people across Australia.

Coghlan's estimate is over 5 times the population enumerated in 1891. Is it just a guess, or can it be given weight as a 'best estimate'? In his 1894 edition, Coghlan writes:

At the census of 1891, only 38,879 aborigines were enumerated, of whom 8,280 were in New South Wales, 565 in Victoria, 23,789 in South Australia, and 6,245 in Western Australia. The figures relating to New South Wales and Victoria include all aborigines now living in those provinces, but the returns from the other Colonies are very imperfect. The aborigines of Tasmania are extinct, but the Tasmanian Census of 1891 enumerates 139 half-castes, which are included in the general population. It has been asserted that there are some 70,000 Aborigines in Queensland. This is, however, a very crude estimate, and may be far wide of the truth. In the case of South

Australia, a large number of the aborigines in the Northern Territory are entirely outside the bounds of settlement, and it seems probable that they are as numerous in that Colony as in Queensland. The census of Western Australia includes only those aborigines in the employment of the colonists, and as large portions of this, the greatest in area of all the Australasian Colonies, are as yet unexplored, it may be presumed that the number of aborigines enumerated in the census is very far short of the total in the Colony. The aboriginal population of the entire Continent may be set down at something like 200,000.

In 1904, based on the 1901 census, which had enumerated 48,248 Aborigines (40,880 full blood, 7,368 half caste), Coghlan (1904 Yearbook) gave an estimate of 153,000. This estimate comprised the enumerated 'full blood' and 'half caste' populations in NSW, Victoria and Tasmania, with estimates of 25,000 for Queensland, 50,000 for SA and 70,000 for WA. He stressed that, except in Victoria, Tasmania and New South Wales, the census enumerations 'must not be taken as indicating the strength of the aboriginal population'.

The first Australian Yearbook in 1908 gave an estimate of 150,000 as a 'rough approximation' of the Aboriginal population without reference to Coghlan or any alternative explanation.

The Australian Constitution had provided that:

In reckoning the numbers of the people of the Commonwealth, or of a State or other part of the Commonwealth, aboriginal natives shall not be counted (section 127).

The background to this Section of the Constitution has not been well described. Relevant material is included in the Appendix.

Section 127 applied to population estimates, not to the scope of the census itself. So, despite the general belief, 'aboriginal natives' were counted in the 1911 census and all subsequent censuses. The count included those

in the employ of whites or (who) were living in contiguity to the settlements of whites.

A total of 19,939 were enumerated in 1911 (Yearbook 1913), compared to 40,880 in 1901. As well, 10,113 of half or lesser Aboriginal blood were enumerated (who were included in the population count), compared to 7,368 in 1901, giving a total enumerated population of 30,052.

It is clear that the Commonwealth Statistician did not regard the census as giving a sound estimate of the Aboriginal population. The 1913 Yearbook discusses the possible population of 'aboriginal natives'. It reports that

various guesses...have been made, and the general opinion appears to have prevailed that 150,000 might be taken as a rough approximation to the total.

It then gives estimates by State from two Queensland Chief Protectors of Aborigines (some of which are stated as 'at least' ...) and concludes:

In view of these figures, it would appear that the number of aboriginal natives in Australia may be said to be not more than 100,000. The whole matter, however, is involved in considerable doubt. (P108).

The race is said to be 'remarkable and rapidly disappearing' (P107). These estimates and associated text are repeated in each Yearbook until 1919.

In the 1920 Yearbook, the 150,000 figure is still mentioned, as are the 'considerable doubt' and 'remarkable and rapidly disappearing' comments. Again based on Queensland Protector estimates, it is concluded that

the number of full-blood Australian aboriginals has been less than 80,000 for several years.

This is not inconsistent with the 1913 estimate of 100,000 aboriginal natives which would have included those considered 'half castes'.

So the Commonwealth Statistician gives an imprimatur over several years to the view that the Aboriginal population is 100,000 rather than 150,000, based on estimates he considers to be subject to considerable doubt. And he repeatedly comments that the race is 'rapidly disappearing'.

The 1920 text is repeated for several years. But a new source is introduced, estimates provided by State Chief Protectors of Aboriginals. With an adjustment in respect of WA, these total 58,771. This leads to a quite definitive conclusion in the 1921 Yearbook that 'it would seem that the marshalling of every aboriginal of full blood in Australia would fail to muster a total of 60,000 at the present time.' Surprisingly, this number reappears in the 1994 Yearbook: 'In the years following colonisation the Aboriginal and Torres Strait Islander population declined dramatically under the impact of new diseases, repressive and often brutal treatment, dispossession, and social and cultural disruption and disintegration. Such data as is available suggests that the Aboriginal and Torres Strait Islander population had declined to around 60,000 by the 1920s.' Remarkably, the 'full blood' qualification of 1921 is omitted.

In 1923, when the 1921 census results are available, adding the census count for 'half caste' Aboriginals to the Chief Protectors' estimates for 'full blood' people gives a total population of 70,273.

From 1924 to 1944, the Bureau undertook an annual Aboriginal census. The results are reported in the Yearbooks from 1925 to 1946-47. The total numbers fluctuate in the range 70,000-80,000. The 'half caste' figure doubles from around 12,000 to 25,000 (around 35% of the total) over the 20 years, and the 'full blood' estimates fall from over 60,000 to 47,000. The results were presented each year without commentary.

These annual censuses in fact continued the approach used by the Statistician in 1920, namely relying on the estimates of State Protectors. Possible undercount, and its variation over time, was not addressed. Smith (1980) discusses how, at various times, some people of aboriginal origin, but less than 50% of Aboriginal blood, were classed as 'of European race', in other words a clear source of undercount. Given the stigma surrounding aboriginality, there would also be incentives for people of aboriginal origin not to reveal it. Another issue arises from the

policy of removal of mixed race children from their families: there was a strong incentive not to inform government of the birth of a mixed race child. Finally, at this time, there were still people living in remote areas beyond the reach of the Protectors.

4.2 1945 to 1967

Following the 1947 and 1954 censuses, the Bureau of Census and Statistics estimated the Aboriginal population by adding the estimated number of 'full' blood people estimated by welfare authorities to the census counts of 'half caste' people. In 1947, the total was 73,817 excluding 5,000 Torres Strait Islanders, giving a total of 78,817 (1953 yearbook). In 1954, there were 70,678 excluding Torres Strait Islanders, who were not separately enumerated.

Following the 1961 census, the Aboriginal population estimate is based on census counts plus adjustments for WA and the NT for people 'out of contact at census'. The 1967 Yearbook (P 206) estimates the Aboriginal population at 30 June 1961 at 79,253, of whom 39,172 met the criteria for inclusion in national population counts, and 40,081 were 'full-blood'. In addition, there were 4972 'full-blood' Torres Strait Islanders and 245 'half European blood' Torres Strait Islanders, giving a total Indigenous population of 84,270.

After the 1966 census, the enumerated population is reported. This totalled 80,207, excluding 5,403 Torres Strait islanders, a total of 85,610.

By the 1960s, there was pressure to change the constitutional provisions in regard to Aboriginal and Torres Strait Islander peoples. A 1944 referendum seeking a range of additional powers for the Commonwealth had included the 'power to make laws (for) people of the aboriginal race. This had been defeated. But, after extensive lobbying by Aboriginal people and others, a referendum to allow the Commonwealth to make laws for Aboriginal and Torres Strait islander peoples was passed in 1967, with a Yes vote of over 90%.

Of specific relevance here, Section 127 was repealed at that same referendum.

4.3 1967 to 1990

The 1967 referendum and the repeal of clause 127 of the Constitution freed the Bureau of Census and Statistics from its requirement to enumerate full blood Aboriginal people separately from other Aboriginal people, so that the former could be excluded from counts of the population. So new ways of identifying and enumerating Aboriginal people could be considered.

Jones (1970) had reported on the impact of changing methods of identifying Aboriginal people on census counts, and the great uncertainty involved. In particular, Jones points out that 1966 counts were not consistent with those of 1961 because the definition had changed so that ‘many thousands of part-Aborigines who were counted as “half-caste” in 1961 described themselves as less than half-Aboriginal in 1966, and were consequently classified with the non-Aboriginal population’. He proposed that the time had come to attempt enumeration of ‘Aborigines as genetically and socially defined’ (his emphasis). He notes that that such an approach was being advocated by the newly established (Commonwealth) Office of Aboriginal Affairs. (While Jones’ book was published in 1970, the Preface makes it clear that publication had been delayed, so presumably Jones’ views were known to those in the late 1960s who were planning the 1971 Census.)

Rowse (2006) has pointed out that one outcome of the 1967 referendum could have been that Aboriginality would no longer be included in the census. He presents the evidence from the 1960s that there was a strong push led by the Social Science Research Council of Australia’s multi-author project ‘Aborigines in Australian Society’ that there needed to be positive discrimination in favour of Indigenous people to overcome persistent disadvantage, and to do that, much improved statistics on Indigenous people were needed. He describes the period 1966-1976 as ‘watershed years’ in Indigenous statistics in Australia.

In the 1966 census, the census question had been as follows:

State person’s race. For persons of European race wherever born, write European

Otherwise state whether Aboriginal, Chinese, Indian, Japanese, etc., as the case may be.

*If of more than one race give particulars, for example,
 $\frac{1}{2}$ European- $\frac{1}{2}$ Aboriginal, $\frac{3}{4}$ Aboriginal- $\frac{1}{4}$ Chinese,
 $\frac{1}{2}$ European- $\frac{1}{2}$ Chinese.*

In 1971, it was quite different:

What is this person's racial origin? (If of mixed origin indicate the one to which he considers himself to belong)

(Tick box only or give origin only)

1 European origin

2 Aboriginal origin

3 Torres Strait Islander origin

4 Other origin (give only)

In 1981, the question was refined, and has remained in much the same form up to the present:

For persons of mixed origin, indicate the one to which they consider themselves to belong.

Is the person of Aboriginal or Torres Strait Islander origin ?

No..... 1

Yes, Aboriginal..... 2

Yes, Torres Strait Islander..... 3

From the 1971 census onwards, the reported Aboriginal population rose rapidly. The numbers were reported without comment in the Yearbooks. There was no effort to restate the estimates of earlier years.

In the 1970s, large scale removal of Aboriginal and Torres Strait Islander children slowed, and according to some, ceased. Moreover, stigma surrounding Aboriginality decreased after the 1967 referendum and the emergence of specific national policies for Indigenous people, such as the Northern Territory Land Rights Act of 1976. So some of the possible sources of undercount begin to diminish over this period. But, as Gray (2002) succinctly comments, 'Successive censuses often yield

population sizes which are evidently incompatible...The inescapable conclusion is that censuses capture an incomplete portion of the Indigenous population each time'.

While the population estimates began to rise after 1971, the Yearbooks for many years contain no discussion or analysis. Importantly, there is no re-examination of the low estimates of earlier years. The 1981 Yearbook, for example, states that 'by 1933, (the Aboriginal population) is estimated to have totalled about 67,000'; this is despite the fact that the 1933 Aboriginal census estimated a population of 79,568. As late as 1989, the Yearbook reports that the population had 'dwindle(d)' to about 60,000 after the first 100 years (the estimate is unsourced).

Smith (1980) attempted to bring all the information together into a set of population estimates for various years from 1788 up to 1971 (Table 4, which also includes various estimates published by official statisticians). He recognised the problem of under identification and that the published figures were too low, and so included estimates of 'Minimum Total Population' from 1788 to 1971. These estimates are 'minimum size ancestral to those *currently* identifiable' (emphasis added). His estimates are below 100,000 from 1901 to 1947 but reach 150,000 in 1971 compared to the official 116,000. Importantly, Smith's estimates do not attempt to include people of Aboriginal descent who were counted in the census but not counted as Aboriginal. He notes that 'there is a sizeable number of persons of Aboriginal or Islander descent in Queensland who do not identify at the census'. Therefore it is to be expected that estimates for earlier years based on 2001 patterns of identification would be higher than Smith's.

4.4 A new approach in the 1990s

Indigenous population issues were not comprehensively addressed by the ABS until the late 1990s. Ross (1999) explored the increase in Indigenous people counted in recent censuses, including the extent to which growth exceeded that expected due to births and deaths, referred to as 'errors of closure'. This work set the scene for ABS estimates of the Indigenous population, which were first published in 1998. These were labelled experimental, a tag still applied to the 2004 estimates. The

continued use of the term experimental after 6 years is unfortunate. The methods employed are the best available given the uncertainties around census counts, births and deaths, and methodology. They are subject to error and revision, as are all ABS estimates.

Neither Ross' nor later publications have sought to estimate the population in earlier years, or to speculate about the sources of error in earlier estimates. Ross lists all the census enumeration counts and estimates from 1911 without comment.

In 1994, the ABS commenced publishing Experimental Estimates of the Aboriginal and Torres Strait Islander Population, based on census results (ABS 1994). These have included estimates for the most recent and the preceding census year. Subsequent publications have appeared in 1998 and 2004 (ABS 1998, 2004).

The 2004 ABS estimate for the Indigenous population in 2001 is 459,000. The ABS estimates for 1996 and 1991 published in 2004 are 414,000 and 361,000 respectively, whereas the 1996 and 1991 estimates published in 1998 were 386,000 and 345,000. Contrasting the 1998 and 2004 ABS estimates highlights the need to restate earlier estimates as new information becomes available.

An estimate (517,000) for 2006 was published by ABS in 2008 (ABS 2008).

(The 'plausible scenario' population estimates in this paper are based on information published up to and including the 2006 estimate. In 2009, the ABS published experimental life expectancy estimates based on the 2006 census using a new methodology (ABS 2009a). Subsequently, ABS has published revised population estimates for 1986 to 2001 based on the new life expectancy methods (ABS 2009b). These methods have been and continue to be the subject of discussion amongst a range of experts. A revision to the 'plausible scenario' may be considered in the light of the new ABS work.)

5 Recent Partial Population Investigations

In recent years, the populations of some States and Territories have been reviewed in some detail. Smith et al (2007) have reported on work in Victoria using the Koori Health Research Database (KHRD). He concludes that there was a substantial population of 'invisible' Aborigines from 1890.

The KHRD data now provides the first direct evidence of this: in Victoria, the census population began to fall below the population ancestral to today's Aborigines in the 1890s, the very time when the policy of forced assimilation was being implemented; that gap widened over time as more and more people of mixed descent joined the 'invisible' group.

Condon et al (2004) have used available official statistics for the Northern Territory (NT) from 1966 and a plausible algorithm (using information only available in the NT) for identifying Indigenous people in the years before identification was included in births and deaths statistics (pre 1988) to produce an internally consistent population time series for the NT from 1966 to 2001. They note that the estimates are consistent with the ABS experimental estimates 1991-1996, based on the 1996 census.

They conclude:

The back-cast method produced an internally consistent time-series from 1966 to 2001, which is in close agreement with the ABS Experimental Estimates for 1991-1996. The back-cast population estimates for 1966 and 1971 were close to the estimates of the NT Administration for those years.

The back-cast estimates are consistently greater than census counts, with the proportional increase being greater in earlier years (14%-40% in 1966-1986 compared to 12% in 1996 and 2001).

Both the Victorian and NT analyses support the hypothesis that the census, at various time periods, has systematically under-estimated the actual Indigenous population.

6 Towards a Consistent Time Series for the Indigenous Population

Recent ABS estimates of the Indigenous population are based on an intensive enumeration effort in successive population censuses. The census data is collected in an environment where the earlier sources of undercount have diminished in importance, although they may not have entirely disappeared. But the ABS has not back cast its estimates based on the 2001 census beyond 1991 to produce a consistent time series which fits plausibly with these recent estimates.

Over time, as already shown, definitions of Indigenous people have evolved, the scope of statistical collections and estimates have varied, and enumeration methods have changed dramatically. It would seem impossible to allow in a detailed way for these changes.

Nevertheless, there is a vast 'Indigenous statistical archive' (Rowse 2006). To make sense of it over time, a consistent view of Indigenous needs to be adopted. The estimates below treat current enumeration practice as the base, and in fact use ABS estimates from 1991 onwards. So Indigenous people are included in the estimates based on the current self identification criteria, using multiple inputs from available statistics and analyses.

6.1 Estimates of Population Growth Rate

A growth rate from 2001 to 2006 cannot be calculated as yet because the estimates for the two years are not on a comparable census base. The annual growth rate from 1996 (414,390) to 2001 (458,520) was 2.04%.

From 1991 (366,943) to 1996, the growth rate was 2.46%. An earlier estimate is available for the 1991 to 1996 growth rate: Kinfu and Taylor (2002) analysed census data and concluded that the 'explained' annual growth rate (the growth rate after removing the effects of enumeration and identification changes) between the 1991 and 1996 censuses was 2.3%.

Based on the 1994 estimates, the growth rate from 1986 to 1991 is 2.45%.

ABS estimates of population growth rates before 1986 do not exist, and cannot be calculated directly. The absence of reliable birth and death statistics prevents estimation of the ‘error of closure’, the impact of changed census methods and possible changes in identification, from the available census data. Kinfu and Taylor report the sharp falls in Indigenous fertility between 1970 and 1980 and the more gradual decline from then on. Adult mortality has not changed significantly in the years for which reliable data is available for WA, SA and the NT (since 1990), but infant mortality fell sharply up to 1980 before levelling out.

Smith (1980) discusses age distribution and population growth rate in some detail. He identifies a progressive change in the age distribution which occurred at different times in different States. He identifies 6 stages, A to F. The age distribution moves progressively from Stages A-C to Stage D, beginning at the 1901 census for NSW (1891 for Victoria, which has a much smaller population than NSW), and for the other States (except NT) at the 1933 census. Stages A-C are consistent with falling or constant population, Stage D with ‘moderate’ growth. Stage E, ‘high’ growth, commences at the 1947 census in NSW, 1961 in other States but not till 1966 for the NT. A ‘population explosion’ stage F is noticed at the 1966 census in NSW and Victoria, but reverts to Stage E for all jurisdictions in 1971.

Smith (1980) also estimated population growth rates up to 1971, which are shown in Table 1.

7 A Plausible Scenario

What can be concluded about the actual Aboriginal and Torres Strait Islander population since Federation? Definitions have varied over time, and data sources are deficient until recent times. So precision is not possible. Estimates should as far as possible reflect the definition used today, which relies on self identification. The extent to which the sources used below reflect this is inevitably a matter for judgement. Certainly different concepts such as enumerated population will give too low an estimate.

7.1 1891-1971

We have seen that Coghlan gave a considered estimate of 200,000 in the 1890s, reduced to 150,000 in the early 1900s. The Commonwealth Bureau of Census and Statistics repeated Coghlan's estimate, and recognised it as a 'general opinion'. But then based on estimates from State protectors, a lower estimate was adopted, despite 'considerable doubt'. A plausible scenario is a 1901 population figure of 150,000. This estimate for 1901 is substantially higher than Smith's 1980 Minimum estimate for that year (94,654), but it should be remembered that his estimates refer to a population ancestral to those currently identifiable in 1971. (Interestingly, Smith projects aboriginal population for 2001 on two scenarios. His higher estimate is 285,415, compared to the actual ABS estimate of 458,520).

In 1901, the year of Federation, the Australian population is estimated to have been 3,788,120. The Aboriginal estimate of 150,000 is 4% of the total Australian population. The estimate exceeds the enumerated population of 48,248 by some 100,000. Given the small number of mixed blood people enumerated in 1901, it is likely that only a negligible number of the 100,000 would have been enumerated but not identified as Aboriginal. This raises the interesting issue of the possible need to restate the Australian population in 1901 and earlier years and related statistics (eg, Million Milestones, Yearbook 2001)

Smith estimates a decline of 20% in his Minimum estimates between 1901 and 1921. Applying this percentage to the plausible 1901 estimate of 150,000 would give a plausible estimate for 1921 of 120,000.

Based on the change in age distributions over the period 1901-1921 reported by Smith, the rate of population decline was decreasing between 1901 and 1921, so an estimate of 130,000 for 1911 is plausible.

The years from 1921 to 1947 need to be considered in the face of the undercount issues discussed earlier. Based on the constancy of the published estimates from 1921 to 1944, a possible course would be to assume a constant population over these years. An alternative, and preferred approach, is to apply Smith's growth figures to produce estimates from 1920 onwards. The result is an estimate bottoming out in

1933 at 117,000, then rising to 138,000 in 1947.

A similar approach can be used up to 1971, where Smith's growth rate series stops. The 1971 population estimate resulting is 238,000. The intermediate years are shown in Table 1, which summarises the estimates described from 1891 to 1971.

Table 1: Estimates up to 1971

Year	Plausible scenario
1891	200,000
1901	150,000
1911	130,000
1921	120,000
1933	117,000
1947	138,000
1954	159,000
1961	186,000
1966	210,000
1971	238,000

7.2 1986-2006

From 1986 to 1991, and from 1991 to 2001, growth rates can be derived from the various ABS Experimental estimates of Indigenous population, and are shown in Table 4. The 2006 estimate is currently a stand alone number. The 2001 estimates include back cast estimates for 1991 and 1996. But for growth between 1986 and 1991, only the 1994 experimental estimates are available on a consistent basis to allow a growth rate to be calculated; the estimated annual growth rate from 1986 to 1991 is 2.45%.

The population estimates for 1991, 1996 and 2001 are on a consistent basis. A 1986 estimate can best be derived by using the growth rate mentioned above to backcast the 1991 population estimate published in 2004 to 1986. This gives a 1986 estimate of 325,117.

7.3 Back casting from 1986

Smith reported a growth rate between 1966 and 1971 of 2.53%, and noted that the rate was slowing (Stage F age distribution no longer applied in NSW and Victoria). For the longer period from 1954 to 1971, Smith's annual growth rate is approximately 2.4%.

From 1986 to 1996, the ABS derived growth rates are approximately 2.45%.

These rates suggest that to back cast from 1986, a growth rate of 2.45% over the period 1971 to 1986 seems plausible. This gives estimates (rounded to the nearest thousand) of 288,000 for 1981, 255,000 for 1976 and 226,000 for 1971.

The back cast estimates from 1971 to 2006 are shown in Table 2 shows actual ABS estimates from 2006 back to 1991, and back cast estimates from 1986 to 1971.

Table 2: Estimates back to 1971

Year	Plausible scenario: Back cast
2006	517,043
2001	458,520
1996	414,390
1991	366,943
1986	325,117
1981	288,000
1976	255,000
1971	226,000

These very different approaches needless to say do not immediately give a smooth time series. The estimate for 1971 using the data available for years up to 1971 is 238,000. The back cast estimate for 1971 (from the data available for years from 1986 onwards) is 226,000. The difference between the estimates is 12,000, or around 5%.

The data from 1986 onwards, while always labelled experimental by the ABS, and which is not all on a consistent basis, should be given most weight in reconciling the estimates. It is based on modern concepts of Indigenous identification, and results from a significant effort in each census to enumerate the Indigenous population.

The authors have therefore chosen to project the 1971 back cast estimate to 1947, using Smith's growth rates (which are shown in Table 4 below). The results are shown in Table 3.

Table 3: Back cast estimates 1971 to 1947

Year	Back cast estimates (Base year 1971)
1971	226,000
1966	199,000
1961	178,000
1954	151,000
1947	132,000

The 1947 back cast estimate differs from the estimate in Table 1 by 6,000. The lower estimate has been adopted for 1947. No further refinement of earlier years has been attempted, given the lack of precision in the estimates available in the earlier years.

8 Summary: A Consistent Plausible Scenario

The following restatement results from the material presented in this paper:

- At the time of European settlement, there were over 750,000-800,000 Aboriginal people in Australia
- Numbers fell during the following century, as a result of disease and dispossession, to some 200,000 in the 1890s
- At Federation, Aboriginal numbers were hard to estimate accurately, but based on contemporary commentary, an estimate of 150,000 is plausible.

- The population continued to decline until about 1920. The published estimates are subject to a series of sources of undercount that cannot be allowed for in a precise way. A figure of 120,000 around 1920 is plausible
- The population reached its minimum in 1933, estimated at 117,000
- The population then began to increase, at first gradually (estimated at 132,000 in 1947) and then more quickly, to reach 226,000 in 1971 and 367,000 in 1991
- In 2006, the population was estimated by the ABS to be 517,000, around 2.5% of the population

The time series from 1788 to 2006 is shown in Table 4.

Table 4: Estimates of the Aboriginal Population

Year	Plausible scenario	Official view (as discussed unless sourced)	Smith (1980) Minimum estimates	Growth rates (from previous value)
1788	750-800,000	300,000-1,000,000	314,500	
1861			180,402	-0.0076 ⁸
1871			155,285	-0.0150 ⁸
1881			131,666	-0.0165 ⁸
1891	200,000	200,000	110,919	-0.0171 ⁸
1901	150,000	150,000	94,564	-0.0160 ⁸
1911	130,000	100,000	83,588	-0.0123 ⁸
1921	120,000	80,000	75,604	-0.0100 ⁸
1933	117,000	70-80,000	73,828	-0.0019 ⁸
1947	132,000	78,817	87,000	0.0117 ⁸
1954	151,000	70,678 ^(a)	100,048	0.0200 ⁸
1961	178,000	84,270	117,495	0.0230 ⁸
1966	199,000	85,610	132,219	0.0236 ⁸
1971	226,000	115,953 ^(b)	150,076	0.0253 ⁸
1976	255,000	160,915 ^(b)		
1981	288,000	159,807 ^(b)		
1986	325,117	250,738 ¹¹		
1991	366,943	366,943 ¹²		0.0245
1996	414,390	414,390 ¹²		0.0246
2001	458,520	458,520 ¹²		0.0204
2006	517,043	517,043 ¹³		NA

^{a)} Excludes Torres Strait Islander people in Queensland,

^{b)} Census enumeration

9 Conclusion

The material set out in this paper shows the historical record concerning population estimates for Australia's Indigenous peoples from 1788 to the present is full of inconsistencies, and is coloured by the social and political views about Indigenous people at various times.

It is inevitable that Indigenous population estimates will never be precise. The data sources that would permit more precise estimates prior to recent years (reliable census counts, good reporting of births and deaths, consistent identification) do not exist. But recognition that past estimates have systematically been too low, right up until the present time, and substitution of a plausible set based on all available data, both contemporary and recent, would provide a better basis for discussion of the historical position of Indigenous people in Australia, the effects of European settlement and the impact of various policies applied to Aboriginal people over the years.

Appendix

The Australian Constitution and Aboriginal Population

At Federation, the Constitution provided that ‘In reckoning the numbers of the people of the Commonwealth, or of a State or other part of the Commonwealth, aboriginal natives shall not be counted’ (section 127). The reason for this clause merits exploration.

It had been agreed during the 1891 Constitutional Convention in Sydney that population, not the number of electors, should determine the allocation amongst the States of the seats in the House of Representatives (which were to be, and for the States still are, fixed at twice the number of seats in the Senate). But how would the population be determined for this purpose. The numbers of one group in particular, namely Aborigines, were basically unknown. There was a second, distinct issue about excluding from the population count of a State for apportionment purposes the numbers of any group who were denied the franchise in that State.

Towards the close of the 1891 Convention, Sir Samuel Griffith (Queensland) put forward the clause excluding ‘aboriginal natives’ from the population in determining estimates of the population (National Australian Convention Debates, 1891). Souter (1988) describes Griffith, who was to become Australia’s first Chief Justice: ‘The real leader (of the 1891 convention) was the Premier of Queensland, Sir Samuel Griffith. This “lean, ascetic, cold, clear” lawyer, as Deakin described him, was elected chairman of the committee on constitutional powers and functions. He did more than anyone else to shape the draft Constitution of 1891’.

Griffith said this provision had been included in the original draft Constitution Bill to be considered at that Convention, which he and others had composed ‘during Easter on the Queensland Government steamer *Lucinda*, as it cruised the waters of the Hawkesbury estuary, Pittwater and Sydney’s Middle Harbour’ (Souter 1988), just prior to the Convention. For some reason, the relevant clauses had been replaced and the exclusion had been lost, so he was acting to reinstate it. He did

not give any rationale, and his proposal was carried without debate.

In the 1897 Convention, an amendment to the Griffith clause was put that ‘aliens not naturalised’ should be excluded along with aboriginal natives. This amendment was lost, on the basis that the franchise issue and the population issue were distinct. Section 25 of the Constitution provides that if all persons of a race are excluded from the vote in a particular State, the persons of that race shall not be counted in the population of that State for apportionment of members of the House of Representatives. (section 25 remains in the Constitution today). It was used to exclude a total of 26,806 people (in Queensland, SA and WA) from the total population of 3,952,491 (0.7%) in determining quotas for the House of Representatives in 1904 (1904 Yearbook).

But Section 25 applies only if all members of a race are excluded, so it did not deal with unnaturalised aliens as a group. Were the delegates just confused? Or was it that the number of aliens not naturalised could be reasonably estimated, but the number of Aboriginal natives could not. In any case, the original Griffiths wording and intention remained, with some minor wording change, and became section 127.

Smith (1980) includes a statement from Sir Alfred Deakin in the 1897 Convention debates ‘Well, it will take 26,000 (aboriginals) to affect one vote’. Deakin was mistaken. As few as one person can affect the number of House of Representatives seats allocated to a State (recent experience with the quota in the ACT and the NT demonstrates the point well).

Smith (1980) concludes that the inclusion of section 127 was largely symbolic, the various debates around it having been confused. He refers to ‘the role that outright xenophobia and racism played in the Federal movement’. It is clear that the interests of Aboriginal people were not considered in the drafting of the Constitution. But there is a clear enough reason for section 127. As Souter (1988) succinctly puts it, ‘This section had made some sort of sense in the 1890s because of the practical difficulty of enumerating remote Aborigines at that time’.

This explanation is given weight by reference to a speech by Griffith in 1896. 'It is plain that the apportionment of the representatives from each State in this House must be in proportion to population. No other basis has, indeed, ever been seriously suggested. But it does not follow that the gross population should be the basis. In the original Constitution of the United States of America the number for the purpose of apportionment was arrived at by adding to the whole number of free persons, excluding Indians not taxed, three-fifths of all other persons.'

As a Queenslander, his State stood to gain by inclusion of aboriginal natives in the population for allocation of House seats. But he preferred certainty, and so did his colleagues. Clearly the prevailing view of Aboriginal people at the time fitted with this conclusion. But it is more a pragmatic move than just a symbolic one.

Smith (1980) notes that section 127 did not limit in any way the census and statistics power included in section 51(xi) of the Constitution. It was not an instruction not to enumerate aboriginal natives, but an instruction to exclude them from counts of population. But exclusion of the 'aboriginal native' group from population counts sent a very clear subliminal message that an extraordinary effort and cost to include these people (and the cost per person would be much higher, as it remains today) was not expected.

The first Commonwealth Statistician obtained a legal opinion that 'persons of the half blood' are not 'aboriginal natives' for the purposes of the Constitution. So the first (1911) Australian census included in its population counts of the Australian population only those of half or lesser Aboriginal blood: 'The cards relating to full-blooded Australian Aborigines were eliminated' (Statistician's Report.1917)

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Nonparametric Bayesian Credibility

T K Siu and H Yang *

Abstract

This paper introduces nonparametric Bayesian credibility without imposing stringent parametric assumptions on claim distributions. We suppose that a claim distribution associated with an unknown risk characteristic of a policyholder is an unknown parameter vector with infinite dimension. In this way, we incorporate the uncertainty of the functional form of the claim distribution associated with the unknown risk characteristic in calculating credibility premiums. Using the results of Ferguson (1973), formulas of the Bayesian credibility premiums are obtained. The formula for the Bayesian credibility pure premium is a linear combination of the overall mean and the sample mean of the claims. This is consistent with the result in the classical credibility theory. We perform a simulation study for the nonparametric Bayesian credibility pure premiums and compare them with the corresponding Bühlmann credibility premiums. Estimation results for the credibility premiums using Danish fire insurance loss data are presented.

Keywords: *nonparametric Bayesian credibility, risk characteristic of policyholder, random probability distribution, credibility premium principle, Dirichlet process.*

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1 Introduction

Credibility theory is an important and useful method in actuarial science. The idea of credibility may be traced to the works of Mowbray (1914) and Whitney (1917). Bailey (1950) introduced statistical ideas to deal with credibility problems. The classical papers by Bühlmann (1967, 1970) were the first to provide a solid mathematical foundation for credibility and established a model-based approach for credibility based on the least-square estimation. The key idea of the Bühlmann credibility model is the use of a linear approximation, which provides actuaries with a convenient way to evaluate credibility premiums. The Bühlmann credibility premium principle for pure premium is a linear combination of a prior mean of the risk parameter and the sample mean of the insurance claims data, which is easy to interpret and makes the intuition of premium calculation more appealing. Waters (1993) provided an excellent introduction to credibility. For a comprehensive discussion on various developments and methodologies on credibility, see the monograph by Bühlmann and Gisler (2005). With the development of modern statistical methodologies and computational technologies, it is possible to develop some efficient and flexible ways to compute the posterior distributions of risk parameters, which are used for premium calculation.

In performing an analysis of insurance losses, the manual rate is designed to reflect the expected experience of the entire rating class. Due to the non-homogeneity of policyholders' risk behaviors, an insurer should charge less/more to a policyholder if the experience of the policyholder is consistently better/worse than that assumed in the underlying manual rate. In parametric Bayesian credibility approach, one usually chooses a parametric conditional loss distribution for each risk and a parametric prior distribution to describe how the conditional distribution varies across the risk. In Young (1997), a semi-parametric mixture model was used to represent the insurance losses of a portfolio of risks. Young (1997) used nonparametric density estimation to estimate the prior density from the claim data and used the estimated model to calculate the predictive mean of future claims. Young (1998) further discussed the semi-parametric model and calculated the intervals for the predictive means.

In this paper, we consider the nonparametric Bayesian credibility, and using some results of Lo (1999), formulas of credibility premiums are obtained. The formula for the nonparametric Bayesian credibility pure premium is a linear combination of the overall mean and the sample mean of the claims. This is consistent with the result in the classical credibility theory. The nonparametric Bayesian credibility model provides us with a flexible way to estimate premiums without imposing stringent parametric assumptions on the insurance loss data. It is also in line with the Bühlmann distribution-free credibility formula and provides an alternative justification for the Bühlmann credibility from the perspective of nonparametric Bayesian statistics.

The paper is organized as follows. The next section presents the nonparametric Bayesian credibility. Section 3 gives some numerical examples to illustrate the practical implementation of the proposed method. In Section 4, we perform a simulation study for the nonparametric Bayesian credibility premium and compare it with the Bühlmann credibility premium. Estimation results for the credibility premiums using Danish fire insurance loss data are presented in Section 5. We also study the credibility factors in successive periods when the data emerge. The final section suggests some further research topics. All tables and figures are displayed in Appendix.

2 Nonparametric Bayesian Credibility

In this section, we present the main idea of nonparametric Bayesian credibility.

In the conventional Bayesian credibility, the risk parameter vector of the claim distribution, (the distribution of X_k), associated with the unknown risk characteristic of a policyholder is finite-dimensional. The claim distribution is completely specified by a finite-dimensional risk parameter vector. This only allows us to vary certain characteristics of the claim distribution while staying with the same functional form. In the nonparametric Bayesian credibility, we view the claim distribution associated with the unknown risk characteristic of a policyholder as an unknown infinite-dimensional parameter vector. In this way, we incorporate the uncertainty of the functional form of the claim

distribution associated with the unknown risk characteristic of a policyholder. Firstly, we give an introduction to the nonparametric Bayesian framework. For excellent references of the nonparametric Bayesian statistics, see Dey, Müller and Sinha (1998) and Lo (1999). We suppose that the random distribution function H is an independent increment stochastic process $\{H(t) | t \in [0, \infty)\}$ which increases from zero to one as t runs from zero to infinity; that is,

1. $H(t)$ increases as t does.
2. For any n and time points $t_1, t_2, \dots, t_n \in [0, \infty)$ with $t_1 < t_2 < \dots < t_n$ $H(t_2) - H(t_1), H(t_3) - H(t_2), \dots, H(t_n) - H(t_{n-1})$ are independent random variables.

We use the function H to characterize the uncertainty of the claim distribution for a policyholder. We assume that conditional on $H = g$, where g is an increasing function of t increasing from zero to one, the probability law for X_1, X_2, \dots, X_{n+1} is given as follows:

1. $X_1, X_2, \dots, X_{n+1} | H = g$ are conditionally independent and identically distributed.
2. For each $k = 1, 2, \dots, n, n+1$, $X_k | H = g \sim F(t | g)$, where $F(t | g) = P_r(X_k \leq t | H = g)$ is the sampling distribution.

To derive analytical formulas for credibility premiums, we consider a conjugate-prior case. We assume that $H = \{H(t) | t \in [0, \infty)\}$ is a Dirichlet process with shape $\alpha^m G^m = \{\alpha^m G^m(t) | t \in [0, \infty)\}$ associated with the *prior belief* $m \in M$. For simplicity, we state this assumption by $H \sim D(dH | \alpha^m G^m)$ according to the notation used in Lo (1999). Further, we assume that $H(t)$ is a *random* distribution function. The parameter of the probability law of $H(t)$ is the Dirichlet process with parameter $\alpha^m G^m(t)$, where α^m is a positive real number and $G^m(t)$ is a given continuous cumulative distribution function. In practice, there are basically three possible methods to determine α^m and $G^m(t)$. One may determine them from the empirical distribution of the prior claims data from other policyholders having similar risk characteristics. They can also be determined purely from the subjective view, or judgment, of an insurer. Based on the empirical Bayesian statistical method, one can also

use the historical claims data of the policyholder to estimate α^m and $G^m(t)$.

In credibility theory, one is interested in estimating the pure premium $E(X_i | H = g)$ associated with a given risk characteristic described by g . In the context of Bayesian credibility theory, the predictive mean $E(X_{n+1} | x_1, x_2, \dots, x_n)$ of the future claim X_{n+1} given the past claims data x_1, x_2, \dots, x_n is adopted to estimate the unknown, or random pure premium, $E(X_i | H)$. Here, we also employ the predictive mean to estimate the pure premium. In our setting, the unknown pure premium $E(X_i | H)$ associated with the risk characteristic described by H is given by:

$$E(X_i | H) = \int_0^\infty t dF(t | H) . \tag{2.1}$$

Note that given $H = g$,

$$E(X_i | H = g) = \int_0^\infty t dF(t | g) .$$

Let $g^m(t)$ be the probability density function, where $g^m(t) := \frac{dG^m(t)}{dt}$. By double expectation, we can also evaluate the unconditional mean, or the overall mean, of X_i as follows:

$$E(X_i) = E[E(X_i | H)] = E\left[\int_0^\infty t dF(t | H)\right] = \int_0^\infty t g^m(t) dt . \tag{2.2}$$

Our model provides practitioners with flexibility to choose $G^m(t)$ as any proper distribution function. However, when we choose $G^m(t)$ in the context of credibility theory, we need to ensure that the overall mean $E(X_i)$ exists, (ie. $E(X_i) < \infty$), so that the credibility pure premium, which depends on the overall mean $E(X_i)$, makes sense. For example, it is not reasonable to assume that $G^m(t)$ is a Cauchy distribution in the estimation of credibility pure premium since the tails of a Cauchy distribution are so heavy that $E(X_i)$ does not exist.

Note that the parameter α^m can be interpreted as the strength of our prior belief on G^m . The larger α^m is, the stronger prior belief on G^m is. This function of α^m becomes more clear when we look at the

posterior distribution of H .

From some standard results, the posterior distribution of $H | \mathbf{x}_n$ is $D(dH | \alpha_{\mathbf{x}_n}^m G_{\mathbf{x}_n}^m)$, where $\alpha_{\mathbf{x}_n}^m G_{\mathbf{x}_n}^m(t) = \alpha^m G^m(t) + \sum_{k=1}^n I\{x_k \leq t\}$. Here we have:

$$\alpha_{\mathbf{x}_n}^m = \alpha^m + n \quad , \quad (2.3)$$

and

$$G_{\mathbf{x}_n}^m(t) = \frac{\alpha^m}{\alpha^m + n} G^m(t) + \frac{n}{\alpha^m + n} \sum_{k=1}^n \frac{1}{n} I\{x_k \leq t\} \quad . \quad (2.4)$$

For the proof, interested readers may refer to Ferguson (1973) and Lo (1999). In the parametric Bayesian credibility, one uses the predictive mean of X_{n+1} given \mathbf{x}_n associated with the prior belief $m \in M$ to approximate the pure premium $E(X | \theta_0)$, where θ_0 is the “true” parameter vector of the claim distribution associated with the risk characteristic of a policyholder. The Bayesian estimate is the minimizer of the expected squared-error loss, (see Bühlmann (1967)). Under some regularity conditions, it can be shown that the predictive distribution of X_{n+1} given \mathbf{x}_n associated with a prior belief $m \in M$ converges to the distribution of X_{n+1} given θ_0 as n tends to infinity, (see Walker (1969)). This implies that

$$\lim_{n \rightarrow \infty} E^m(X_{n+1} | \mathbf{x}_n) = E(X_{n+1} | \theta_0) \quad , \quad (2.5)$$

almost surely.

Note that the convergence result in (2.5) is valid irrespective of the choice of the prior distribution for Θ . For each $m \in M$, $E^m(X_{n+1} | \mathbf{x}_n)$ is called a Bayesian premium associated with the prior belief m .

In our framework, we also use the Bayesian premium associated with m to estimate the pure premium $E(X | g_0)$, where g_0 is the true claim distribution associated with risk characteristic of a policyholder. Then, from Ferguson (1973) and Lo (1999), the nonparametric Bayesian

credibility premium for pure premium associated with $m \in M$ is given as follows:

$$\begin{aligned}
 P(X_{n+1} | \mathbf{x}_n) &:= E^m(X_{n+1} | \mathbf{x}_n) = (1 - Z_n^m) \int_0^\infty t g^m(t) dt + Z_n^m \bar{x}_n \\
 &= (1 - Z_n^m) E(X_i) + Z_n^m \bar{x}_n, \tag{2.6}
 \end{aligned}$$

where $Z_n^m = \frac{n}{\alpha^m + n}$.

The predictive mean $P(X_{n+1} | \mathbf{x}_n)$ is used to estimate the pure premium $E(X_i | H = g)$ associated with the risk characteristic described by g . Note that $P(X_{n+1} | \mathbf{x}_n)$ is a linear combination of the overall mean $E(X_i) := \int_0^\infty t g^m(t) dt$ and the sample mean $\bar{x}_n := \frac{1}{n} \sum_{k=1}^n x_k$ of the claims data with weight Z_n^m . This is consistent with the credibility estimate of the pure premium in the parametric Bayesian credibility.

In fact, the nonparametric Bayesian premium can be related to the Bühlmann credibility premium with appropriate choice of the Dirichlet process. We interpret the weight Z_n^m as the credibility factor placed on the claim data. If the number of data n becomes large, more weight will be placed on the sample mean. If α^m is large compared with n , more weight will be given to the overall mean which is interpreted as the collective premium associated with $m \in M$. α^m can be interpreted as a measure of the confidence for the prior guess at g_0 measured in terms of the number of observations it represents. As α^m tends to zero, the Dirichlet prior is “non-informative”.

For more discussions about the properties of Dirichlet process priors, interested readers may refer to Antoniak (1974), Ferguson (1973) and Lo (1999). The Bayesian premium in (2.6) is again the minimizer of the expected squared-error loss, (see Ferguson (1973)). Further, from the observation that $Z_n^m \rightarrow 1$ as $n \rightarrow \infty$, the predictive distribution of X_{n+1} given \mathbf{x}_n associated with $m \in M$ converges uniformly with probability one to the true claim distribution g_0 as n tends to infinity, (see Ferguson (1973)), where the predictive distribution is given by:

$$F_{X_{n+1}}^m(x | \mathbf{x}_n) = (1 - Z_n^m) G^m(x) + Z_n^m \hat{F}(x) . \tag{2.7}$$

This implies that

$$\lim_{n \rightarrow \infty} E^m(X_{n+1} | \mathbf{x}_n) = E(X | g_0) , \tag{2.8}$$

almost surely, where $E^m(X_{n+1} | \mathbf{x}_n)$ is given by (2.6) and

$$E^m(X | g_0) = \int_0^\infty t g_0(t) dt .$$

Note that $G^m(x)$ is the prior guess on the shape of the unknown $g_0(x)$ based on the prior belief $m \in M$ and $\hat{F}(x)$ is the empirical distribution defined by $\frac{1}{n} \sum_{k=1}^n I\{x_k \leq x\}$. The prior guess that $G^m(x)$ is the prior mean of the Dirichlet process H with shape $\alpha^m G^m$ associated with the prior belief m . It can be shown that the prior mean $G^m(x)$ is the minimizer of the expected squared-error loss, (see Ferguson (1973)). We can determine the prior mean $G^m(x)$ associated with the prior belief m , which reflects an insurer's prior belief, or experience, about the uncertainty of the risk characteristic. The convergence result in (2.8) is valid for each prior belief $m \in M$.

Besides the Bayesian premium principle in (2.6), other more conservative premium principles can be defined by adding a variety of risk loadings which may depend on higher moments of the predictive distribution given in (2.7). For the calculations of the risk premiums involving higher moments, we apply the following general formula which appeared in Lo (1999):

$$E^m(g(X_{n+1}) | \mathbf{x}_n) = (1 - Z_n^m) \int_0^\infty g(t) g^m(t) dt + Z_n^m \int_0^\infty g(t) \hat{F}(t) . \tag{2.9}$$

For each $k = 1, 2, \dots$, we calculate the k -th moment of the predictive distribution given in (2.7) by letting $g(t) = t^k$ and substituting it into (2.9). In particular, we can calculate the squared-loss error for evaluating the estimation error of credibility premiums.

Remarks:

1. One feature of using the nonparametric Bayesian approach is that a claim distribution sampling from a Dirichlet process is discrete with probability one. However, the nonparametric Bayesian approach is particularly useful in calculating credibility premiums involving

higher moments of predictive distributions. This approach provides analytical formulas for both the best estimates of the higher moments of the true claim distribution and their corresponding precisions, and, hence it allows more flexibility in the variations of credibility premium principles. Furthermore, this approach provides a way to estimate the median and quantiles of the true claim distribution.

- Two methods of improving the problem stated in Remark (1) have been proposed in the statistical literature. One of the methods is to find another prior process which chooses a continuous claim distribution with probability one, (see Ferguson (1973)). Another method is to apply the Bayesian mixture model for density estimation introduced by Lo (1984).

We can also evaluate the credibility premium principles involving the first two moments as follows:

- The expected value principle with risk loading coefficient $\lambda_1 > 0$:

$$P_1^m(X_{n+1} | \mathbf{x}_n) = E^m(X_{n+1} | \mathbf{x}_n) + \lambda_1 E^m(X_{n+1} | \mathbf{x}_n) . \tag{2.10}$$

- The standard derivation principle with risk loading coefficient $\lambda_2 > 0$:

$$P_2^m(X_{n+1} | \mathbf{x}_n) = E^m(X_{n+1} | \mathbf{x}_n) + \lambda_2 \sigma^m(X_{n+1} | \mathbf{x}_n) . \tag{2.11}$$

- The variance principle with risk loading coefficient $\lambda_3 > 0$:

$$P_3^m(X_{n+1} | \mathbf{x}_n) = E^m(X_{n+1} | \mathbf{x}_n) + \lambda_3 [\sigma^m(X_{n+1} | \mathbf{x}_n)]^2 , \tag{2.12}$$

where

$$\sigma^m(X_{n+1} | \mathbf{x}_n) = \sqrt{E^m(X_{n+1}^2 | \mathbf{x}_n) - [E^m(X_{n+1} | \mathbf{x}_n)]^2} ,$$

and

$$E^m(X_{n+1}^2 | \mathbf{x}_n) = (1 - Z_n^m) \int_0^\infty t^2 g^m(t) dt + Z_n^m \frac{1}{n} \sum_{i=1}^n X_i^2 .$$

3 Numerical Examples

We provide some numerical examples to illustrate how to implement the proposed nonparametric Bayesian credibility in practice. These examples are based on a combination of real data and hypothetical data. Indeed, the analytical formulae for credibility premiums developed based on the proposed method can be implemented easily via hand calculators. The proposed method satisfies the simplicity requirement for credibility as emphasized by Bühlmann and Gisler (2005).

To illustrate the proposed method, we consider a non-life actuary who wishes to evaluate credibility premiums of automobile insurance. The actuary was given data from the collective, generated by Swiss automobile drivers from the year 1961, which were available to F. Bischsl. The data are shown in the following table.

Table 1: Data of Swiss automobile drivers from the year 1961

No. of Claims	0	1	2	3	4	5	6	Total
No. of Policies	103,704	14,057	1,766	255	45	6	2	119,853

The following table gives the hypothetical claims frequency data for an individual policyholder from 1962 to 1969.

Table 2: Hypothetical claims data of an individual policyholder

Year	1962	1963	1964	1965	1966	1967	1968	1969
No. of Claims	0	1	0	0	1	0	0	1

To simplify our discussion, we assume that the amount of each individual's claim is 5,000. Based on the given claims data, the actuary wishes to evaluate credibility premiums based on the nonparametric Bayesian credibility premium principles presented in the last section. These calculations are discussed in the following examples.

Example 3.1:

Suppose the actuary wishes to evaluate the nonparametric Bayesian credibility premium for pure premium. The actuary needs to evaluate three items, namely, the overall mean, the sample mean and the credibility factor.

Based on the real data on Table 1, the overall mean can be evaluated as follows:

$$\begin{aligned}
 E[X_j] &= \int_0^{\infty} t g^m(t) dt \\
 &= \sum_{k=1}^6 5000k \times \left(\frac{\text{No. of Policies with } k \text{ claims}}{119,853} \right) \\
 &= 774.9493.
 \end{aligned}$$

Note that the prior belief m of the actuary is given by the manual rate based on the real data in year 1961. Unlike the Bühlmann credibility premium principle illustrated in Example 16.26 of Klugman, Panjer and Wilmot (2004), stringent assumptions about the parametric form of the prior distribution of the risk parameter of the individual policyholder are not required to compute the overall mean. This makes the proposed model easy to implement in practice and avoids model misidentification of the prior distribution of the risk parameter.

Based on the hypothetical data on Table 2, the sample mean is given by:

$$\begin{aligned}
 \bar{x}_8 &= \frac{5000 \times (0+1+0+0+1+0+0+1)}{8} \\
 &= 1875.
 \end{aligned}$$

Recall that the parameter α^m is a measure of the confidence on the prior belief which is measured in terms of the number of observations it represents. Suppose the actuary thinks that the information from the manual rate should be weighted as two observations. Then $\alpha^m = 2$, so the credibility factor Z_8^m is evaluated as:

$$Z_8^m = \frac{8}{2+8} = 0.8.$$

The nonparametric Bayesian credibility premium for pure premium associated with the manual rate m is:

$$P(X_9 | \mathbf{x}_8) = (1 - 0.8) \times 1875 + 0.8 \times 774.9493 \\ = 1654.99.$$

Based the above data, the actuary can evaluate risk premiums involving higher moments using the following formula for the j^{th} moment of the predictive distribution of X_9^j given \mathbf{x}_8 .

$$E^m(X_9^j | \mathbf{x}_8) \\ = 0.2 \times \sum_{k=1}^6 (5000k)^j \times \left(\frac{\text{No. of Policies with } k \text{ claims}}{119,853} \right) \\ + 0.8 \times \frac{5000^j \times (0^j + 1^j + 0^j + 0^j + 1^j + 0^j + 0^j + 1^j)}{8} \\ = 0.2 \times \sum_{k=1}^6 (5000k)^j \times \left(\frac{\text{No. of Policies with } k \text{ claims}}{119,853} \right) + 0.3 \times (5000)^j.$$

The formula can be used to evaluate the three credibility premium principles involving the first two moments in Section 2. This is illustrated in the following example.

Example 3.2:

Suppose the actuary wishes to incorporate risk loadings in premium calculations. The actuary may evaluate premiums based on the three principles, namely, the expected value principle, the standard deviation principle and the variance principle. From a theoretical and practical perspective, it seems that the standard deviation principle is more appropriate than the variance premium since the latter involves the squares of monetary units which are difficult to interpret.

Here we calculate premiums based on the expected value premium and the standard deviation principle. We suppose that $\lambda_1 = \lambda_2 = 0.05$. Since the variance principle involves a variance term in the risk loading, one may wish to use a smaller value for the coefficient of risk loading in order to get a reasonable and realistic value for the premium.

Using the expected value principle, the nonparametric Bayesian credibility premium associated with the manual rate m is given by:

$$\begin{aligned}
 P_1^m(X_9 | \mathbf{x}_8) &= (1 + 0.05) \times 1654.99 \\
 &= 1737.7394.
 \end{aligned}$$

Now, we evaluate the second moment of the predictive distribution of the claim as follows:

$$\begin{aligned}
 &E^m(X_9^2 | \mathbf{x}_8) \\
 &= 0.2 \times \sum_{k=1}^6 (5000k)^2 \times \left(\frac{\text{No. of Policies with } k \text{ claims}}{119,853} \right) + 0.3 \times (5000)^2 \\
 &= 0.2 \times 5080807.322 + 0.3 \times (5000)^2 \\
 &= 8516161.464.
 \end{aligned}$$

The standard deviation of the predictive distribution of the claim is:

$$\begin{aligned}
 \sigma^m(X_9 | \mathbf{x}_8) &= \sqrt{8516161.464 - 1654.99^2} \\
 &= 2403.5744.
 \end{aligned}$$

Consequently, using the standard deviation principle, the nonparametric Bayesian credibility premium associated with the manual rate m is:

$$\begin{aligned}
 P_2^m(X_9 | \mathbf{x}_8) &= 1654.99 + 0.05 \times 2403.5744 \\
 &= 1775.1686.
 \end{aligned}$$

4 Simulation Results

We provide a simulation study on the credibility pure premium from the nonparametric Bayesian model and compare it with the Bühlmann credibility premium. We describe the simulation procedures for the risk parameters and the corresponding claims data in the sequel.

Step 1 Simulate 10 risk parameters from a prior distribution $\pi(\theta)$ which is supposed to be Uniform $[1, 10]$, an uniform distribution on the interval $[0, 1]$.

Step 2 Given a fixed risk parameter θ , sample X_1, X_2, \dots, X_{100} from the conditional distribution $\pi(x | \theta)$ of the claim amount given the risk parameter θ , which is supposed to be Pareto $(3, \theta)$, a Pareto distribution with shape parameter $a_0 = 3$ and mode parameter θ .

We suppose that the Pareto distribution with $a_0 = 3$ and θ is the “true” claim data model corresponding to the individual with risk parameter θ and consider the corresponding simulated data as if there were the “true” observations in our simulation experiment. In this way, the “true” pure premium is $E(X_i | a_0 = 3, \theta) = \theta \times \frac{3}{3-1} = 1.5\theta$. Then, we evaluate the Bühlmann credibility premiums B under the Poisson-gamma (PG) model and the Pareto-Uniform (PU) model, and the estimated pure premium P from the nonparametric Bayesian model using the simulated data with 100 observations for each fixed risk parameter. We shall compare the Bühlmann credibility premium B and the estimated pure premium P from the nonparametric Bayesian model with the “true” pure premium. We also calculate the estimated pure premium from the non-parametric Bayesian models and the Bühlmann premiums under the PG model and the PU model and compare them with the “true” pure premiums in successive periods.

Firstly, we suppose that G^m is a Gamma distribution with shape parameter $\alpha = 10$ and scale parameter $\beta = 2$. To compute the Bühlmann credibility premium, we use the Poisson-Gamma model in Example 16.26 of Klugman, Panjer and Willmot (2004). We assume that the prior parameters of the Gamma distribution are $(a, b) = (10, 2)$. Then, the Bühlmann credibility premium under the Poisson-Gamma model is given by:

$$B_{PG}(X_{n+1} | \mathbf{x}_n) = \left(\frac{b}{n+b} \right) \frac{a}{b} + \left(\frac{n}{n+b} \right) \bar{x}_n . \tag{4.1}$$

We also consider the Bühlmann credibility premium under the Pareto-Uniform (PU) model. We suppose that the prior parameters of the Uniform distribution are $(L, U) = (1, 10)$ and the shape parameter of a Pareto distribution $a = 3$. The Bühlmann credibility premium under the Pareto-Uniform model is given by:

$$B_{PU}(X_{n+1} | \mathbf{x}_n) = \left(\frac{k}{n+k} \right) \frac{L+U}{2} + \left(\frac{n}{n+k} \right) \bar{x}_n , \tag{4.2}$$

where

$$k = \frac{4}{(a-1)(a-2)} \left(1 + 3 \frac{UL}{(U-L)^2} \right) . \tag{4.3}$$

Table 3: Estimated Premiums v.s. the “True” Pure Premiums

Case	1	2	3	4	5	6	7	8	9	10
θ	2.8842	7.4075	1.9461	6.5859	5.3743	3.028	3.1345	9.7325	8.6013	4.0759
True	4.3262	11.1113	2.9191	9.8789	8.0614	4.5419	4.7017	14.5988	12.9019	6.1139
Premium Principles	Estimated Premiums									
$B_{PG}(X_{n+1} \mathbf{x}_n)$	4.3816	11.861	2.8717	9.5966	7.9305	4.5198	4.5892	16.2473	14.01	5.9601
$B_{PU}(X_{n+1} \mathbf{x}_n)$	4.3994	11.8249	2.9004	9.5768	7.9227	4.5366	4.6055	16.1795	13.9584	5.9665
$P^n(X_{n+1} \mathbf{x}_n)$	4.3755	11.929	2.8506	9.6421	7.9596	4.5151	4.5851	16.3586	14.0992	5.9696

Table 3 presents the “true” pure premiums, the Bühlmann credibility premiums under the PG model and the PU model, and the nonparametric Bayesian pure premiums using the full simulated data for the 10 risk parameters.

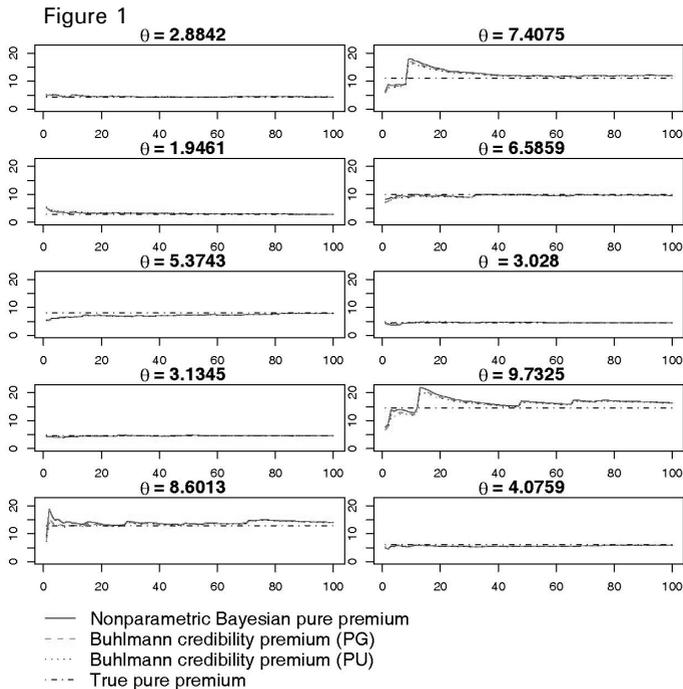


Figure 1 presents the plots the “true” pure premiums, the Bühlmann credibility premiums under the PG model and the PU model, and the nonparametric Bayesian pure premiums using the simulated data in successive periods for the 10 risk parameters.

5 Estimation Results

In this section, we provide estimation results for the premium principles outlined in Section 2 using Danish fire insurance loss data from years 1988 to 1990. It consists of 663 fire insurance losses in Danish Krone (DKK). The data were downloaded from <http://www.ma.hw.ac.uk/~mcneil/ftp/DanishData.txt> (Date: 1 October 2009).

Firstly, we evaluate the premiums from the Bühlmann credibility principles under the PG model and the PU model, the premiums P , P_1 , P_2 and P_3 from the nonparametric Bayesian model using full data. We also investigate the credibility factors of the Bühlmann credibility model and the nonparametric Bayesian model in successive periods. Waters (1993) also provided the study of the credibility factors from the parametric Bayesian model as the claims data emerge.

Here, we adopt the prior parameters for the nonparametric Bayesian model and the Bühlmann credibility models described in the last section to evaluate the premiums from the nonparametric Bayesian model and the Bühlmann credibility premiums. We evaluate the Bühlmann credibility premiums from the PG model and the PU model as in the last section. We consider a set of specimen values of the risk loadings and suppose that $\lambda_1 = \lambda_2 = \lambda_3 = 0.5$.

Table 4: Estimated Premiums

Premium Principles	Estimated Premiums (in Millions)
$B_{PG}(X_{n+1} \mathbf{x}_n)$	3.7091
$B_{PU}(X_{n+1} \mathbf{x}_n)$	3.7126
$P^m(X_{n+1} \mathbf{x}_n)$	3.7072
$P_1^m(X_{n+1} \mathbf{x}_n)$	5.5608
$P_2^m(X_{n+1} \mathbf{x}_n)$	8.3563
$P_3^m(X_{n+1} \mathbf{x}_n)$	46.9358

Table 4 presents the estimates of premiums based on various premium principles using full data, with 663 observations.

Figure 2

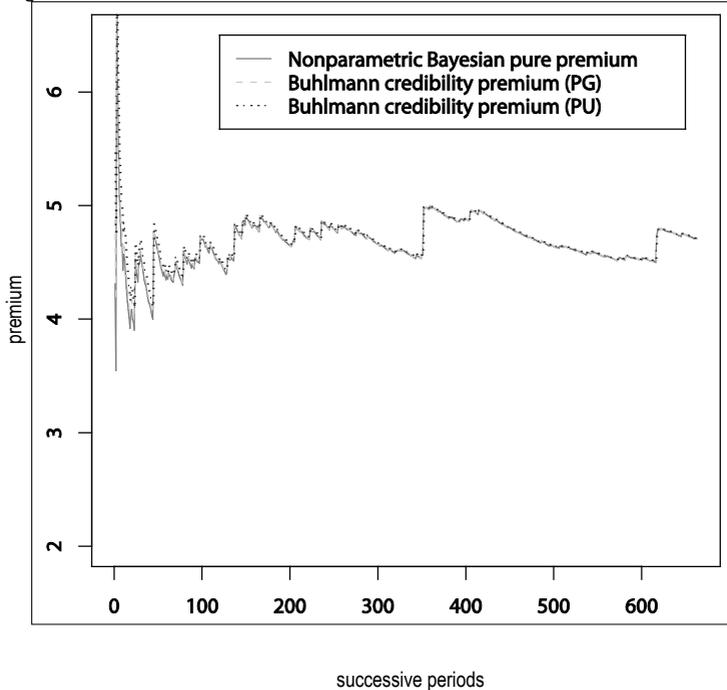


Figure 2 presents the plots of the Bühlmann credibility premium and the nonparametric Bayesian pure premiums in successive periods.

Figure 3

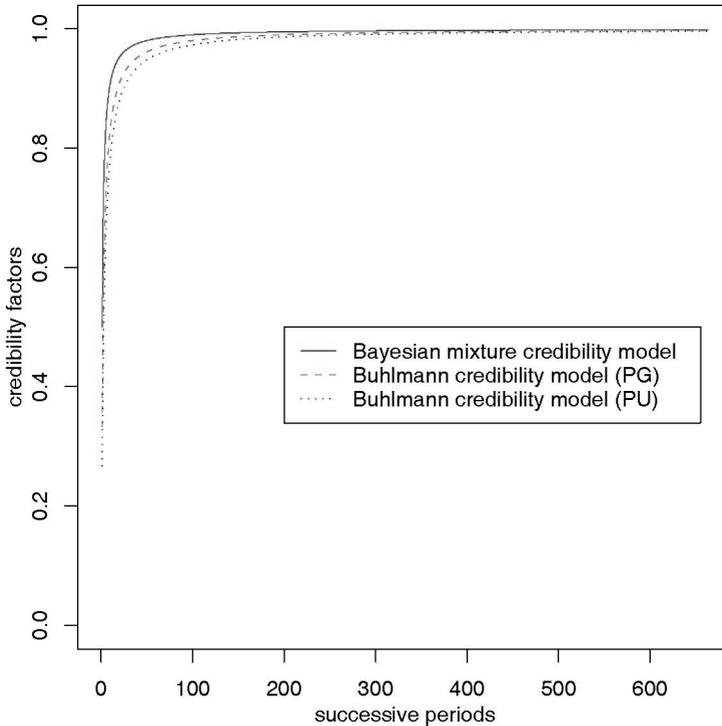


Figure 3 presents the plots of the credibility factors $Z_n = \frac{n}{\alpha^m + n}$ from the nonparametric Bayesian model, credibility factors from the Bühlmann credibility models under the PG model and the PU model in successive periods.

It is of practical interest to conduct robustness analysis with respect to the prior processes and the methods for choosing the shape of the Dirichlet prior process when the sample size is small. Using the techniques from neural networks, more efficient methods of estimating the shape of the Dirichlet prior process are expected to be developed. Analytical formulas for credibility premiums involving the median, quantiles and higher moments of the predictive distribution for the next year claim given the claim data in the past few years can be obtained in our credibility framework, and hence, our credibility

framework provides grounds for the further development of various credibility premium principles. It would be interesting to investigate the credibility premium principles, such as, the premium principle involving the first four accumulations by Ramsay (1994), the principle with the third central moment by Chu (1999) and the risk-adjusted credibility premiums with distorted probabilities by Wang and Young (1998), and others.

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Comparing Discretisations of the Libor Market Model in the Spot Measure

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Abstract

Various drift approximations for the displaced-diffusion LIBOR market model in the spot measure are compared. The advantages, disadvantages and implementation choices for each of predictor-corrector and the Glasserman-Zhao method are discussed. Numerical tests are carried out and we conclude that the predictor-corrector method is superior.

Keywords: LIBOR market model, BGM, drift approximation, spot measure, predictor-corrector, martingale discretisation, Glasserman-Zhao, TARN

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1 Introduction

It has become more common in recent years for insurance companies to invest their reserves in exotic structured products in order to obtain an increase in yield. Such products typically consist of a note issued by a bank which pays a coupon which is a path-dependent function of LIBOR or swap rates. This note is often callable, that is the bank can repay the principal early and terminate the contract. It is therefore important to be able to accurately assess the value of these products.

Methodologies for pricing such interest-rate products have evolved over time. The post Black–Scholes models for pricing interest rate derivatives were initially short rate models. These models relied on the notion of a hypothetical short rate chosen to follow a process that was typically some variant of a normal or log-normal process.

One can view exotic options pricing as an extrapolation exercise, the objective being to extrapolate from the prices of vanilla derivative instruments such as swaps and bonds, and options thereon, to prices of exotic derivatives. One therefore had to fit the short rate process's parameters to the prices of market instruments. In particular, calibrating to both the discount curve and the prices of swaptions were non-trivial tasks. A consequence of all this fitting and the fact that everything was driven by a single short rate was that the dynamics of the model were not realistic. The interest rate discount curve can evolve in a complicated fashion which is not capturable by a single state variable.

The viewpoint later shifted to evolving market observable forward rates and in particular, the LIBOR market model has become popular. Its biggest virtue is the ability to easily calibrate to the discount curve and to caplet volatilities whilst still having enough flexibility to calibrate to other instruments and to specify correlations. Although it is now over ten years old, various details of its implementation have still not been fully resolved. In this paper, we focus on one of these: the best way to approximate the stochastic differential equation (SDE) evolution when using the discretely compounding money market account as numeraire.

Unlike in the Black–Scholes model, the stochastic differential equation describing the evolution of the forward rates involves a state-dependent drift. This means that there is no analytic solution to the SDE and it must be numerically approximated. A numerical approximation corresponds to a choice of discrete time model, we will compare two such approximations in this paper: the predictor-corrector approximation of Hunter, Jäckel and Joshi [10] and the “arbitrage-free” discretisation of Glasserman and Zhao [9]. Whilst there have been many other papers written on the topic of drift approximations in the LIBOR market model these focus on approximations when using the final bond as numeraire: [1], [7], [17], [15], [21], [22]. Here we are interested exclusively in the case where the numeraire is the discretely compounding money market account. Our motivation is that the variance of a Monte Carlo pricing simulation is generally lower when working with this numeraire, see for example [3]. No comparison between these two popular methods appears to have been done previously.

This issue is of importance to practitioners as they want a Monte Carlo simulation which has low variance whilst suffering from only small discretisation errors. Whilst discretisation errors can always be reduced by taking extra steps, this will inevitably increase the time taken and it is therefore an undesirable solution.

We structure the paper as follows. In Section 2, the displaced-diffusion version of the LIBOR Market Model is introduced, with corresponding discretisation schemes using both predictor corrector and Glasserman-Zhao methods. In Section 3, we present a number of issues with the most popular Glasserman-Zhao discretisations in any measure other than the terminal measure. Numerical results for forward rate agreements (FRAs), caplets, digital caplets, and target redemption notes (TARNs) are given in Section 4, and we conclude in Section 5.

2 Displaced-diffusion LIBOR Market Model

We recall the set-up of the LIBOR market model which is also often known as the BGM (Brace–Gatarek–Musielka) model or the BGM/J (Jamshidian) model. For more detail, we refer the reader to the

fundamental papers [4],[12], [18] and [20], or to the books [3], [5], [8], [14], [19] and [23]. The basic idea is to evolve discrete market-observable forward rates, rather than hidden unobservable factors. We have tenor dates $0 < T_0 < T_1 < \dots < T_n$, with corresponding forward rates f_0, \dots, f_{n-1} , so that f_i is the forward rate for the period T_i to T_{i+1} . Set $\tau_i = T_{i+1} - T_i$.

Let $P(t, T)$ denote the price at time t of a zero-coupon bond paying one at its maturity, T . This paper primarily focuses on using the *spot LIBOR measure*, which corresponds to using the discretely-compounded money market account as numeraire, within the LIBOR market model. This numeraire is made up of an initial portfolio of one zero-coupon bond expiring at time T_0 , with the proceeds received when each bond expires being reinvested in bonds expiring at the next tenor date, up until T_n . More formally, the value of the numeraire portfolio at time t will be,

$$N(t) = P(t, \eta(t)) \prod_{i=0}^{\eta(t)-1} (1 + \tau_i f_i(T_i)),$$

where $\eta(t)$ is the unique integer satisfying

$$T_{\eta(t)-1} \leq t < T_{\eta(t)},$$

and thus gives the index of the next forward rate to reset. The use of the discretely compounded money market account as numeraire within the LIBOR market model was introduced in [12]. When using this numeraire we will have to step to each forward rate reset time during a simulation because of the path dependent nature of the numeraire (see Remark 2 of [21]).

Under the displaced-diffusion LIBOR market model, the forward rates f_i are assumed to have the following evolution

$$df_i(t) = \mu_i(f, t)(f_i(t) + \alpha_i)dt + \sigma_i(t)(f_i(t) + \alpha_i)dW(t), \tag{1}$$

where the σ_i are deterministic F dimensional row vectors, the α_i are constant displacement coefficients, W is a standard F -dimensional Brownian motion under the relevant martingale measure, and the μ_i are determined by the no-arbitrage condition that all bond prices (ie. model tradable assets) discounted by the numeraire must be martingales in the

pricing measure. The σ_i are given by a pseudo square root (or suitable approximation) of the instantaneous covariance matrix for the f_i 's.

Under the spot LIBOR measure, the drift term is given by

$$\mu_i(f, t) = \sum_{j=\eta(t)}^i \frac{(f_j(t) + \alpha_j)\tau_j}{1 + f_j(t)\tau_j} \sigma_i(t)\sigma_j(t)',$$

see [5], where a $'$ is used to indicate the transpose of a matrix.

We consider displaced diffusion, as discussed in [14], [5], because it is a simple way to allow for the skews seen in implied caplet volatilities that have persisted in interest rate markets since the market events of 1998, see [14]. In any case, the results presented here will collapse to the non-displaced case when $\alpha_i = 0$ for all i .

In what follows, we will discuss evolution from time T_i to time T_{i+1} . Working with $\log(f_i + \alpha_i)$ terms is favorable as it yields state-independent volatility. We then have to evolve

$$d \log(f_j(t) + \alpha_j) = \left(\mu_j(f, t) - \frac{1}{2} \sigma_j(t)\sigma_j(t)' \right) dt + \sigma_j(t) dW(t),$$

with the drift term given above.

If drifts were not state-dependent, this equation would have an exact solution and simulation would be easy. Since they are state-dependent, approximations must be used, and as such, a large amount of work has been done on methods to approximate the term

$$\int_{T_i}^{T_{i+1}} \mu_k(f, t) dt, \tag{2}$$

that arises in simulation. The simplest approximation is Euler stepping, which uses

$$\hat{\mu}_k = \sum_{j=i+1}^k \frac{(f_j(T_i) + \alpha_j)\tau_j}{1 + f_j(T_i)\tau_j} \tilde{\sigma}_k(T_i, T_{i+1}) \tilde{\sigma}_j(T_i, T_{i+1})'.$$

to approximate (2), where $\tilde{\sigma}_k(T_i, T_{i+1})$ is defined such that

$$\tilde{\sigma}_k(T_i, T_{i+1}) \tilde{\sigma}_j(T_i, T_{i+1})' = \int_{T_i}^{T_{i+1}} \sigma_k(y) \sigma_j(y)' dy,$$

and thus gives the k^{th} row of a pseudo-square root to the integrated covariance matrix. As is well documented in the literature, $\tilde{\sigma}_k(T_i, T_{i+1})$ is often used instead of $\sigma_k(T_i)$ to make the approximation of (2) more accurate, see [5]. We will use the $\tilde{\sigma}$ notation presented here throughout.

The Euler stepping method involves calculating the drift term assuming the f_j 's are constant between $[T_i, T_{i+1}]$ and equal to their value at the start of the interval. More sophisticated approximations have been suggested in [15], [1], [21], and also ways to get around using such approximations, [9]. We will focus on two of the most popular methods, namely predictor-corrector [10], and the Glasserman-Zhao approach [9], presenting arguments against using the latter method when simulating under any measure other than the terminal measure.

2.1 Predictor-Corrector

The predictor-corrector method was introduced to the LMM in [10] following Kloeden and Platen [16], and is sometimes referred to as the HJJ method after the authors. The idea is to evolve forward rates pretending all state variables are constant, recompute the drift at the evolved time and average the two drifts. We then re-evolve the forward rates using this averaged drift and the same random numbers. Formally, we use

$$\hat{\mu}_k = \frac{1}{2} \sum_{j=i+1}^k \left[\frac{(f_j(T_i) + \alpha_j)\tau_j}{1 + f_j(T_i)\tau_j} + \frac{(\hat{f}_j(T_{i+1}) + \alpha_j)\tau_j}{1 + \hat{f}_j(T_{i+1})\tau_j} \right] \tilde{\sigma}_k(T_i, T_{i+1})\tilde{\sigma}_j(T_i, T_{i+1})'$$

to approximate (2), where a hat above a quantity implies that it has been estimated through an initial log-Euler evolution.

2.2 Glasserman-Zhao

The Glasserman-Zhao method instead uses a change of coordinates to simulate the forward rates. The idea is to map the forward rates to positive martingales. Being driftless and positive, we can use Euler-stepping to evolve the logs of these quantities preserving positivity

and the martingale property. The forward rates can then be recovered from the evolved quantities. This method has the theoretical virtue that deflated bond prices can be made to be martingales even in the discretised model used for simulation. This is not the case for other discretisation methods such as predictor-corrector which provide a very good non-martingale approximation to a martingale measure. The major advantage is that the no-arbitrage condition described in [18] can be made to hold even in our discrete simulation, and as such, the prices of zero-coupon bonds and FRAs can be recovered without bias. On the negative side, while the Glasserman-Zhao method can price up to one caplet according to the Black formula, [2], without bias, there can be significant bias for other caplets, see [15] for the terminal case.

One can consider any number of possible changes of coordinates. However, here we will focus only on the most popular ones, which generally make up what is referred to as the Glasserman-Zhao method. In particular, we consider evolving the logarithm of the difference between adjacent deflated bonds, ie. $\log((P(t, T_j) - P(t, T_{j+1}))/N(t))$, with minor adjustments to ensure positivity of discount ratios and to take account of displacements.

Define

$$D(t, T_j) = \frac{P(t, T_j)}{N(t)}, \tag{3}$$

$$= \prod_{k=0}^{j-1} \frac{1}{1 + \tau_k f_k(t)}. \tag{4}$$

When $\alpha_j = 0$ for all j , we evolve the logs of

$$\begin{aligned} V_j(t) &= (D(t, T_j) - D(t, T_{j+1})), \\ &= \tau_j f_j(t) \prod_{k=0}^j \frac{1}{1 + \tau_k f_k(t)}, \quad j = 0, \dots, n-1, \end{aligned}$$

which are positive martingales in continuous time. Euler stepping the logs of the V_j 's in the simulation ensures each simulated V_j is a positive martingale and, as we shall see later, the recovered $D(\cdot, T_j)$'s are monotonic and decreasing in j , and are martingales. Note that the introduction of displacements means that in the continuous time

specification of the LIBOR market model, forward rates can become negative, and thus discounted bond values need not be monotonic. As such, our discretisation should not try to force this by preserving strict monotonicity, but should instead aim to only allow non-monotonic discount ratios in line with that allowed in the continuous time specification. Thus, to introduce displacements, while maintaining a positive martingale with which to evolve our forward rates, we suggest using

$$V_j(t) = \tau_j(f_j(t) + \alpha_j) \prod_{k=0}^j \frac{1}{1 + \tau_k f_k(t)},$$

$$= D(t, T_j) - (1 - \tau_j \alpha_j) D(t, T_{j+1}),$$

which will remain positive provided $0 \leq \alpha_j \leq \frac{1}{\tau_j}$, as would be the case in any realistic set-up. Using such V 's maintains all the properties that were present in the case of no displacements, except for the possibility of non-monotonic discount ratios, a shortcoming of the displaced diffusion LIBOR market model. This method of incorporating displacements in the Glasserman-Zhao method was originally suggested by Brace, [3].

Using the fact that the V 's must be martingales in the pricing measure, we can derive their stochastic differential equations using the same methods as used in the original Glasserman-Zhao paper. We obtain,

$$\frac{dV_j(t)}{V_j(t)} = \left[\sigma_j(t) - \sum_{k=\eta(t)}^j \phi \left(\frac{V_k(t)}{D(t, T_k)} \right) \sigma_j(t) \right] dW(t), \tag{5}$$

where $\phi(x) = \min\{1, x^+\}$ is necessary to ensure the diffusion coefficient in (5) is Lipschitz-continuous in the discrete case. Equation (5) suggests a possible discretisation with

$$\hat{V}_j(T_{i+1}) = \hat{V}_j(T_i) \exp \left(-\frac{1}{2} \sigma_{\hat{V}_j}(T_i, T_{i+1}) \sigma_{\hat{V}_j}(T_i, T_{i+1}) + \sigma_{\hat{V}_j}(T_i, T_{i+1}) \tilde{Z}_i \right), \tag{6}$$

for $j = i + 1, \dots, n - 1$, where

$$\sigma_{\hat{V}_j}(T_i, T_{i+1}) = \left[\sigma_j(T_i, T_{i+1}) - \sum_{k=i+1}^j \phi \left(\frac{\hat{V}_k(T_i)}{\hat{D}(T_i, T_k)} \right) \sigma_k(T_i, T_{i+1}) \right], \tag{7}$$

where $\tilde{Z}_0, \tilde{Z}_1, \dots$ denote independent F -dimensional standard normal column vectors, and hats are used to indicate simulated quantities. Note that under such a discretisation, it is easy to see that $E_i[\hat{V}_j(T_{i+1})] = \hat{V}_j(T_i)$, where $E_i(\cdot)$ denotes the expectation conditional on the information available at time T_i in the discretised measure; so the simulated V 's will remain positive martingales in the discrete model.

So far we have changed coordinates from forward rates to V_j 's and have simulated a set of V_j 's. All that we need to do now is recover the forward rates to price our products. We do this by recovering the discounted bond prices, which can easily be used to recover the forward rates, from

$$\hat{D}(T_{i+1}, T_j) = \frac{\hat{D}(T_i, T_{i+1}) - \sum_{k=i+1}^{j-1} \hat{V}_k(T_{i+1}) \prod_{l=k+1}^{j-1} (1 - \alpha_l \tau_l)}{\prod_{k=i+1}^{j-1} (1 - \alpha_k \tau_k)}, \tag{8}$$

for $j = i+2, \dots, n$, with the understanding that $\sum_{k=i}^j = 0$ and $\prod_{k=i}^j = 1$ when $i > j$. Note that under the spot LIBOR measure, $D(T_i, T_{i+1}) = D(T_{i+1}, T_{i+1})$ since $D(t, T_{i+1})$ only depends on forward rates which have all reset by time T_i .

Since the \hat{D} 's are recovered as a linear combination of the \hat{V} 's, they will also remain martingales in the discrete simulation. However, as discussed below, using this exact change of coordinates allows the possibility of negative discount ratios.

It is important to realize that negative discount ratios are a much more serious issue than negative interest rates. As long as one does not allow the ability to hold cash in the economy, a negative interest rate does not imply an internal arbitrage in the model. However, a negative value for a zero-coupon bond implies that one can buy something for a negative amount that will have a guaranteed value of 1 in the future; this is clearly an arbitrage internal to the model.

3 Assessing the Glasserman-Zhao discretisation

In this section, we look at the various methods used to overcome the problem of negative discount ratios. We consider two interpretations of the method presented in [9] identifying significant short comings in each. We also consider an obvious way one could attempt to deal with these shortcomings, showing that it too is flawed. While we conduct our discussion in the spot LIBOR measure, it is important to realise that all the problems identified will naturally extend to measures based on hybrid numeraires, ie. where some zero-coupon bond with expiry on one of the inner tenor dates is used as numeraire.

3.1 The terminal measure

Although our focus is on the spot measure, it is worth noting why the issues with negative discount ratios do not arise in the *terminal measure*. For details of the discretisation in the terminal measure see [9].

Under the terminal measure the expression for recovering the deflated bond prices is given by

$$\begin{aligned} \hat{D}_j^{terminal} &= \prod_{k=j}^{n-1} (1 - \alpha_k \tau_k) + \sum_{k=j}^{n-1} \hat{V}_k^{terminal} \prod_{l=j}^{k-1} (1 - \alpha_l \tau_l), \\ &= 1 + \sum_{k=j}^{n-1} \hat{V}_k^{terminal}, \text{ where } \alpha_i = 0 \ \forall i. \end{aligned}$$

Starting from $D_n \equiv 1$ we work backwards, each time adding another positive term to the previous \hat{D} until $\hat{D}(T_{i+1}, T_{i+1})$ is reached. Therefore the $\hat{D}(., T_j)$'s are positive, as well as decreasing in j when displacements are zero so all forward rates will be positive.

3.2 Method 1 - negative discount ratios

As noted in [9], using (8) to recover the \hat{D} terms from the \hat{V} 's can lead to negative \hat{D} 's. The reason this problem arises in the spot LIBOR measure, but not when using an equivalent simulation scheme in the terminal measure is that we are simulating the differences

$$\hat{V}_j = \hat{D}(\cdot, T_j) - (1 - \alpha_j \tau_j) \hat{D}(\cdot, T_{j+1}),$$

but when recovering the \hat{D} 's we now start from $\hat{D}(T_i, T_{i+1})$ and work forward, calculating

$$\hat{D}(\cdot, T_{j+1}) = (\hat{D}(\cdot, T_j) - \hat{V}_j) / (1 - \alpha_j \tau_j)$$

from simulated and already-calculated quantities. This means the \hat{D} 's are recovered as a difference; since there are no appropriate constraints, this can and does lead to negative value for \hat{D}_j .

3.3 Method 2 - backwards method 1

The problem of the previous method stemmed from working forwards in j , and thus requiring the recovered \hat{D} 's be calculated as a difference. An obvious alternative is therefore to find a way in which we can recover the \hat{D} 's by working backwards, starting from $\hat{D}(T_{i+1}, T_n)$. Glasserman-Zhao suggest adding a new term to the simulated quantities,

$$V_n = D_n,$$

with dV_n given by (5) where $\sigma_n(t) = 0$. Trivially, we can also extend (6) and (7) to hold for $j = n$ by imposing $\sigma_n(t) = 0$. Using \hat{V}_n , we can now recover the discounted bond prices by working backwards, giving

$$\hat{D}(T_{i+1}, T_j) = \sum_{k=j}^n \hat{V}_k(T_{i+1}) \prod_{l=j}^{k-1} (1 - \alpha_l \tau_l) \tag{9}$$

$$= \sum_{k=j}^n \hat{V}_k(T_{i+1}), \text{ where } \alpha_i = 0 \forall i. \tag{10}$$

This method ensures the \hat{D} 's remain positive martingales as the \hat{V} 's are positive martingales by construction.

There are two possible ways to use this idea. In this section, we focus on the seemingly theoretically correct interpretation, leaving the practically appealing implementation to the next section. The method discussed here is the one described in [3].

Within the new set-up, $\hat{D}(T_i, T_{i+1}) = \hat{D}(T_{i+1}, T_{i+1})$ is known at the start of the step from T_i to T_{i+1} , and will not change. Thus to recover the unknown \hat{D} 's, using (9), we need to simulate $\hat{V}_{i+2}, \dots, \hat{V}_n$. Importantly, we are now simulating \hat{V}_j for $j = i + 2, \dots, n$ instead of for $j = i + 1, \dots, n - 1$. The problem with this is that the properties of f_{i+1} enter the simulation only in an indirect and minimal way. To see this, consider (6) and (7). The term which does all the work in the simulation is (7), in which θ_0 dominates. As \hat{V}_{i+1} is not simulated in this method, the volatility of the first forward rate not yet reset, σ_{i+1} , does not enter the discretisation in a complete way; it only has a partial influence through the summation term of (7) where all coefficients are less than or equal to one. If, for example, we are pricing a product that is highly sensitive to the first forward rate, such as a TARN with an increasing initial term structure of forward rates, we will not be simulating the important quantities accurately, and are likely to badly mis-price.

In addition, without displacements, there is nothing to ensure that $\hat{D}(T_{i+1}, T_{i+2}) < \hat{D}(T_{i+1}, T_{i+1})$. Since we obtain

$$\hat{f}_{i+1}(T_{i+1}) = \frac{\hat{D}(T_{i+1}, T_{i+1}) / \hat{D}(T_{i+1}, T_{i+2}) - 1}{\tau_{i+1}},$$

negative forward rates can therefore be obtained which is an undesirable feature when displacement is zero.

In terms of the set-up used in [9], where forward rates are recovered directly from the \hat{V} 's, this method can equivalently be seen as using the relation,

$$\hat{D}(T_i, T_{i+1}) = \hat{D}(T_{i+1}, T_{i+1}) = \sum_{k=i+1}^n \hat{V}_k(T_{i+1}) \prod_{l=j}^{k-1} (1 - \alpha_l \tau_l), \tag{11}$$

to determine $\hat{V}_{i+1}(T_{i+1})$, which can then be used to recover the forward rates with the other \hat{V} 's.

3.4 Method 3 - backwards method 2

In order to address the problems highlighted in the previous section, a second more practically appealing interpretation of Glasserman-Zhao's idea can be used. Rather than worry about discount ratios that have already reset (ie. that depend only on forward rates which have already reset), we can simulate \hat{v}_j for $j = i+1, \dots, n$ and recover $\hat{D}(T_{i+1}, T_j)$ for $j = i+1, \dots, n$ using (9). This ensures the properties of each relevant forward rate fully enter the simulation, and that without displacements the relevant \hat{D} 's are monotonic, ensuring all recovered forward rates are positive.

An obvious disadvantage with this method is that we are required to simulate $n+1$ quantities in a model with an n -dimensional state space. This raises other issues. In particular, we are now recalculating $\hat{D}(T_{i+1}, T_{i+1})$ at time T_{i+1} , after it has already reset, and there is nothing to ensure that it will be equal to its value at the start of the step. For example, consider evolving from 0 to T_0 . By definition, $D(T_0, T_0) = 1$. However, under this method we recalculate $\hat{D}(T_0, T_0)$ at time T_0 , but do not ensure that it is equal to 1. A similar effect will occur at each evolution time. As such, additional discretisation bias has been introduced into the method.

On an even more practical level, the additional discretisation bias can cause problems when incorporating this method in pre-existing libraries. In particular, if the determination of payoffs and subsequent discounting are done independently, it is likely $N(T_{i+1})$ will be valued inconsistently with the pay-off, meaning the pricing of FRAs will not be bias-free. This follows since at each time T_{i+1} , the principal in the numeraire portfolio is given by

$$\prod_{j=0}^i (1 + f_j(T_j)\tau_j) \equiv \frac{1}{\hat{D}(T_i, T_{i+1})},$$

which is naturally updated at the end of the previous step from T_{i-1} to T_i (ie. without taking into account the additional bias described above). To see the effect of this, consider pricing the FRA from T_{i+1} to T_{i+2} struck at K within a larger simulation. We evolve to T_{i+1} and calculate the payoff as

$$\left[\frac{\hat{D}(T_{i+1}, T_{i+1}) / \hat{D}(T_{i+1}, T_{i+2}) - 1}{\tau_{i+1}} - K \right] \frac{\hat{D}(T_{i+1}, T_{i+2})}{\hat{D}(T_{i+1}, T_{i+1})}$$

We then divide by the numeraire value, $N(T_{i+1})$. If

$$N(T_{i+1}) = \frac{1}{\hat{D}(T_{i+1}, T_{i+1})}, \tag{12}$$

the discounted payoff will be

$$\frac{\hat{D}(T_{i+1}, T_{i+1}) - \hat{D}(T_{i+1}, T_{i+2})(1 + K\tau_{i+1})}{\tau_{i+1}},$$

and we have no problems since this is a linear combination of martingales in the discrete model, and is therefore also a martingale. However, since additional bias has been introduced to $\hat{D}(T_{i+1}, T_{i+1})$, (12) does not hold, and our FRAs are not priced without bias as the discounted payoff will no longer be a martingale (but instead the ratio of linear combinations of martingales) in the discrete simulation. More fundamentally, our discrete simulation is no longer arbitrage free. An example of a library where this would be an issue is *QuantLib*.

3.5 Method 4 - re-normalisation

To correct for the additional discretisation bias introduced by the previous method, an obvious solution is to use renormalization. In particular, to ensure that $\hat{D}(T_{i+1}, T_{i+1}) = \hat{D}(T_i, T_{i+1})$, we can rescale the \hat{V} 's simulated using (6) and (7). We replace \hat{V}_j with

$$\bar{V}_j(T_{i+1}) = \frac{\bar{D}(T_i, T_{i+1})}{\sum_{k=i+1}^n \hat{V}_k(T_{i+1}) \prod_{l=j}^{k-1} (1 - \alpha_l \tau_l)} \hat{V}_j(T_{i+1}), \tag{13}$$

where the \bar{D} 's are recovered as in (9), but with the \bar{V} 's instead of the \hat{V} 's.

This will guarantee that discount ratios do not change after they have reset, while maintaining the practical advantages of the method presented in the previous section. However, the \bar{V} 's are now recovered as a ratio of linear combinations of the \hat{V} 's, and thus so are the \bar{D} 's.

Since the \hat{V} 's are martingales by construction, the recovered \bar{V} 's and \bar{D} 's will no longer be martingales in the discretised model. While this re-normalisation has given desirable properties, it has destroyed the reason for using Glasserman-Zhao in the first place, which was to evolve forward rates whilst ensuring that discount ratios were martingales.

4 Numerical Results

To compare prices obtained using log-Euler, predictor-corrector, and the four Glasserman-Zhao methods we price FRAs, caplets, digital caplets, and TARNs. As a term structure we take yearly forward rates f_0, f_1, \dots, f_{19} (with the first reset at $t_0 = 1$), all equal to 5%. The instantaneous correlation between rate i and rate j is given by

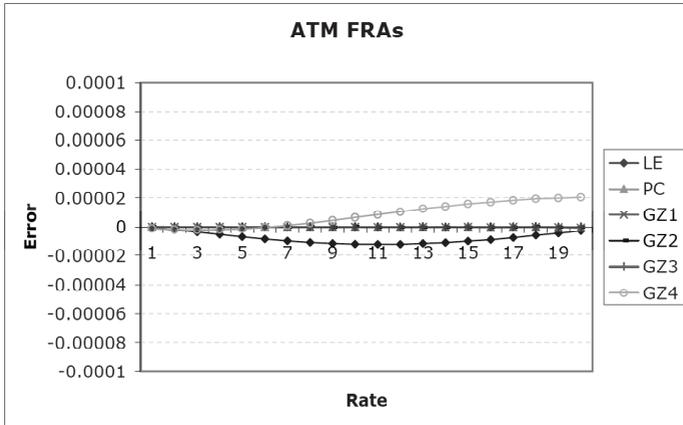
$$\rho_{ij} = e^{-\beta|T_i - T_j|},$$

with $\beta = 0.04$. High correlation (small β) equates to larger drift terms and is therefore a tough test of the different methods. The instantaneous volatility is flat at 15% and the displacement for each forward rate is 1.5%.

We ran 2^{21} paths using Sobol quasi-random numbers with Brownian Bridging (for details of Brownian Bridging see [11]). This large number of paths was chosen to ensure that any errors are mostly due to the method rather than convergence.

Figure 1 shows the pricing errors for an at-the-money (ATM) FRA using log-Euler (LE), predictor-corrector (PC), and the four Glasserman-Zhao methods (GZ1 = Glasserman-Zhao method 1, etc.). The 20 points on the x -axis represent FRAs where the underlying forward rate resets at x years. The y -axis scale ranges from minus one basis point (bp) to 1bp.

Figure 1 Pricing errors for at-the-money FRAs, strike = 0.05



As we expect, the log-Euler method does not price FRAs without error. The two most interesting features are that predictor-corrector prices FRAs without any noticeable error and that the re-normalised Glasserman-Zhao method performs similarly to log-Euler. The error in the re-normalised Glasserman-Zhao method can be attributed to the destruction of the martingale property.

To compare caplet prices, we consider the difference between the Black price and the simulated prices for both ATM, figure 2 and out-of-the-money (OTM) caplets, figure 3. The prices of the ATM caplets rise from 35 basis points to 71 basis points and then fall back to 61 basis points. The errors are therefore on the order of 0.3 percent in relative terms.

The first three OTM caplet prices are $1.95E-5$, $2.1E-4$ and $5.3E-4$ and the final one is worth 33 basis points. The errors are therefore very high in relative terms at the short end. Even at the long end, an error of one basis point is roughly 3 percent.

Similarly with FRAs, predictor-corrector prices both ATM and OTM caplets without any significant error. For the ATM caplets, Glasserman-Zhao methods 1 and 4 have errors of similar magnitude to log-Euler. The best performing Glasserman-Zhao method for ATM caplets is method 3, which involves simulating $n + 1$ quantities. This suggests that

the additional bias introduced by simulating already reset rates is small.

Focusing on OTM caplets, it appears that of the four Glasserman-Zhao methods, method 3 again prices with the smallest error. All four methods, however, perform significantly worse than predictor-corrector and arguably worse than log-Euler.

Figure 2 Pricing errors for at-the-money caplets

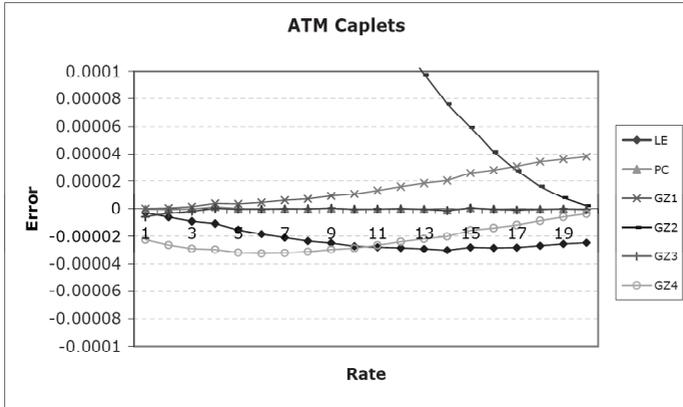
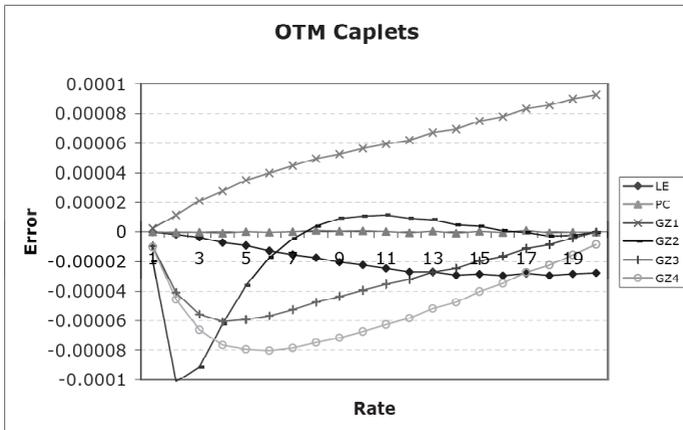


Figure 3 Pricing errors for out-of-the-money caplets, strike = 0.08



We now turn our attention to digital caplets. Figure 4 shows the bias for ATM digital caplets, where the scale ranges from -100bps to 100bps. Both predictor-corrector and log-Euler have only small biases, but the four Glasserman-Zhao methods are very significant biased. The prices of these digital caplets range from 0.4264 (ie. 4264 basis points) to 0.1323. So an error of 100 basis points is on the order of 2.5 percent for the short-dated caplets.

Figure 4 Pricing errors for at-the-money digital caplets

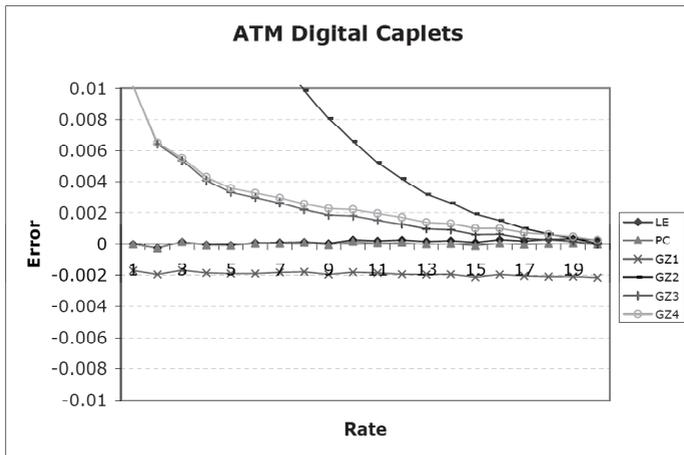
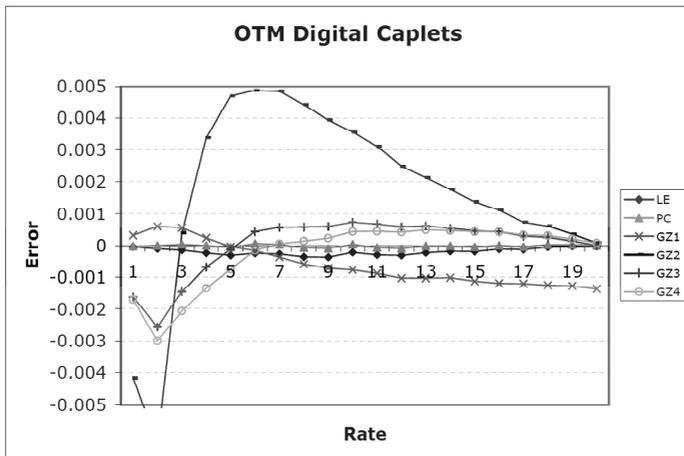


Figure 5 Pricing errors for out-of-the-money digital caplets, strike = 0.08



This bias is also present in OTM digital caplets, where once again log-Euler and predictor-corrector perform better than the four Glasserman-Zhao methods. Products with digital effects provide a stronger test of the approximation methods, since the effect of any bias is magnified significantly.

The prices of the first three OTM digital caplets are 41, 251 and 460 basis points, they then increase to 876 basis points before decreasing to 660 basis points. The errors displayed in Figure 5 are therefore significant in relative terms.

The pricing of digital caplets is an example of this, and the errors suggest that the Glasserman-Zhao methods will not price more exotic products with digital type effects accurately, as we will see later with TARNs.

It is also worth investigating the time taken by each method. Table 1 shows that, for our implementations, predictor-corrector is slower than all the Glasserman-Zhao methods, however the difference is small.

Table 1 Timings from caplet prices for the various methods relative to the time taken for the fastest method: log-Euler.

Method	Time
LE	1.0
PC	1.3
GZ1	1.2
GZ2	1.2
GZ3	1.2
GZ4	1.2

Despite the fact that pricing FRAs, caplets, and digital caplets are good tests, we are ultimately more interested in how the different methods perform for popular exotic products which they are regularly used to price, such as TARNs (for a detailed description of a TARN see [6]). As there is no analytic price for a TARN, we plot prices against step size to see which method converges fastest.

We use a similar setup as before, but with an increasing term structure of forward rates starting from $t_0 = 0$. We have $f_0 = 2\%$ and $f_{20} = 10\%$, with forward rate f_i satisfying, $f_i = \log(a + b \times T_i)$ where the constants a and b are chosen to keep f_0 and f_{20} at their given values.

Figure 6 The price of a TARN varying the number of short steps per step for the six different drift approximation methods.

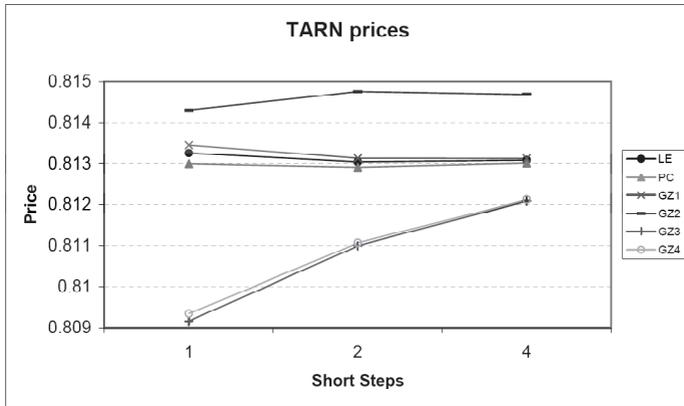


Figure 6 plots the convergence of prices. The TARN pays an inverse floating coupon, $c_i = \max(K - 2f_i, 0)$ with strike, $K = 0.08$ and a total coupon of 0.1. The principal is repaid either at maturity or when the total coupon is reached if that is sooner. The maturity is 20 years. The TARN returns the principal as soon the total coupon is reached. Since the coupons are inverse floating, this means that for an increasing yield curve either the principal is paid back early, or if rates go up then it goes to the full maturity paying no interest. This results in a large change in value according to whether a critical rate is above or below a threshold. It is therefore effectively a digital product.

Similarly to the previous examples predictor-corrector performs very well, converging the fastest. Log-Euler and Glasserman-Zhao method 1 also converge quickly, however, the three other Glasserman-Zhao methods do not perform well at all, which is consistent with the digital caplet examples.

5 Conclusion

In conclusion, although the Glasserman-Zhao method appears to have the theoretical appeal of being an arbitrage-free discretization, we have seen that this is not really the case when working in the spot measure. In practical terms, our numerical results show that in this case predictor-corrector is more effective and we recommend its use, especially for products with a digital effect.

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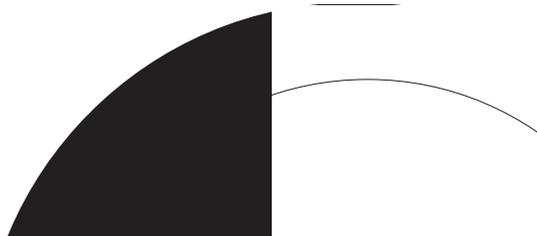
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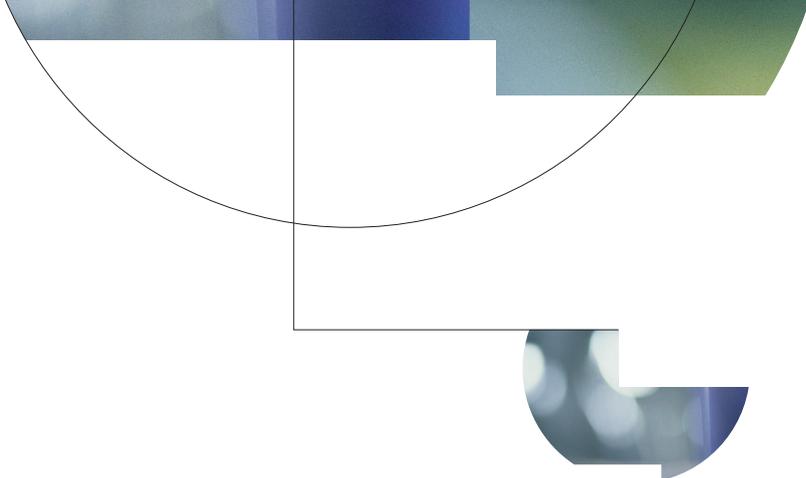
The editor welcomes contributions to the Journal, either as Feature Articles or as Notes. Papers may be practical or theoretical in nature. Full guidelines for authors are available from the Institute of Actuaries of Australia or can be downloaded from the Institute's website www.actuaries.asn.au

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