VALUING ESOs USING THE EXERCISE MULTIPLE APPROACH AND BINARY P-OPTIONS

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Introduction

This paper proposes an analytic formula for valuing Employee Share Options (ESOs) using the exercise multiple approach suggested by Hull (2004) for modelling the early exercise behaviour of holders of ESOs.

ESOs typically have a vesting date, vesting conditions (hurdles) and a period (the exercise window) during which they can be exercised.

Using the exercise multiple approach it is assumed that the option is exercised early as soon as the stock price reaches some multiple of the exercise price. Under this approach, we show that the ESO can be thought of as a combination of non standard barrier options.

We also show this ESO can be expressed as a combination of binary power options using a technique called the method of images.

The binary power option is a building block for the valuation of the ESO. Valuation formulae for these binary power options are presented.

The method of images

This is a general technique that can be used for valuing a wide class of barrier type options.

We discuss the method of images for barrier options and show it can be used to provide valuation formulae for both standard and non standard barrier options, and the relationship of these to ESO valuation.

Barrier Option Features of ESOs

As mentioned above, the Hull White approach to modeling the early exercise behavior of executives with ESOs amounts to assuming the ESO is a form of barrier option.

The hurdles which must be satisfied in order for an option to vest could be defined / measured:

- at a single point in time (the vesting date) or
- over an interval of time (e.g. the vesting period).

If measured over a time interval the ESO would be a barrier type option.

Another type of ESO with barrier type features is the “repriceable ESO” studied by Brenner et al (2000).

This is a type of ESO where, if the stock price declines below some predetermined level before the vesting date, the terms of the ESO are altered so as to increase the value of the ESO.

The strike price and / or the maturity of the ESO can be changed in a predetermined way. This makes the ESO into a type of Barrier Option.

The original ESO is “knocked out” and the new version of the ESO is “knocked in” when the barrier is breached before the original maturity date.

Valuation building blocks

The European power option contract with power n has the payoff at maturity defined as

$V_n(x,T) = f(x)$ where $f(x) = x^n$.

The market value of this contract at time $t < T$ is

$$V_n(x,t) = e^{-r\tau} E_Q\left(f(X_T) \mid X_t = x \right) = x^n e^{\gamma(n)\tau}$$

where

$$\gamma(n) = \frac{1}{2}\sigma^2 n^2 + \left(r - q - \frac{1}{2}\sigma^2 \right) n - r \text{ and } \tau = T - t$$

The **<u>binary power contract</u>** with power n and exercise price K and sign s has payoff at maturity**:** $V_n^s\left(X_T,T\right)=X_T{}^n\mathbb{I}_{\left(sX_T>sK\right)}$

The **<u>up binary power contract</u>** with power n and exercise price K has payoff at maturity**:**
where $s=1,\ V_n^s\left(X_T,T\right)=X_T{}^n\mathbb{I}_{\left(X_T>K\right)}$

The **<u>down binary power contract</u>** with exercise price K has payoff at maturity**:**
where $s=-1, V_n^s\left(X_T,T\right)=X_T{}^n\mathbb{I}_{\left(X_T<K\right)}$
The parameter s takes the values +1 or -1 and controls the direction of the inequality in the indicator function.

The value of these contracts for $s=\pm 1$ at time $t<T$ is given by the formula
$V_n^s\left(x,t\right)=x^n e^{\gamma(n)\tau}N\left(s\left(d_2+n\sigma\sqrt{\tau}\right)\right)$ and, $V_n^+\left(x,t\right)+V_n^-\left(x,t\right)=V_n\left(x,t\right)=x^n e^{\gamma(n)\tau}$

The bond binary option

This has maturity T and
payoff $B_K^s\left(X,T\right)=\mathbb{I}\left(sX>sK\right)$

The value of the contract is $B_K^s\left(x,t\right)=e^{-r\tau}N\left(sd_2\right)$

This is a power binary with power $n=0$, i.e. $B_K^s\left(x,t\right)=V_0^s\left(x,t\right)$
also known as the cash or nothing binary option

The asset binary option

This has maturity T and

payoff $A_K^s\left(X,T\right)=X\mathbb{I}\left(sX>sK\right)$

The value of the contract is $A_K^s\left(x,t\right)=xe^{-q\tau}N\left(sd_1\right)$

This is a power binary option with power $n=1$, i.e. $A_K^s\left(x,t\right)=V_1^s\left(x,t\right)$

Also known as the “asset or nothing binary option”.

Lemma: *Consider the quadratic equation*

$$\gamma(n) = \frac{1}{2}\sigma^2 n^2 + \left(r - q - \frac{1}{2}\sigma^2\right)n - r = 0$$

(i) The solutions of this equation are

$$n = \beta_1 = \frac{-\left(r - q - \frac{1}{2}\sigma^2\right) + \sqrt{D}}{\sigma^2} > 0 \quad n = \beta_2 = \frac{-\left(r - q - \frac{1}{2}\sigma^2\right) - \sqrt{D}}{\sigma^2} < 0$$

where $D = \left(r - q - \frac{1}{2}\sigma^2\right)^2 + 2r\sigma^2$

(ii) The sum of the 2 solutions is $\beta_1 + \beta_2 = -\alpha$ *where* $\alpha = 2\frac{(r-q)}{\sigma^2} - 1$

(iii)*if* $V_\beta(x,t) = x^\beta$ *for* $\beta = \beta_1, \beta_2$ *then* $V_\beta^s(x,t) = x^\beta N\left(s\left(d_2 + \beta\sigma\sqrt{\tau}\right)\right)$

Call, Put and Gap Options

For a European call option the payoff at maturity is $V\left(X_T, T\right) = \left(X_T - K\right)^+$ and this is equivalent to the expression $\left(X_T - K\right)\mathbb{I}\left(X_T > K\right)$

The call option gives the holder the right but not the obligation to buy the stock at maturity for price X and this is worth doing if and only if $\left(X_T > K\right)$ the K here is the "exercise price" which defines the level of the stock at which exercise occurs.

For the call option this is the same as the price to be paid to buy the stock under the terms of the option, which we refer to as the "strike price".

It is convenient to distinguish between the exercise price and the strike price.

<u>A gap call option</u> has a payoff $V\left(X_T,T\right)=\left(X_T-K\right)\mathbb{I}\left(X_T>E\right)$ the exercise price E and the strike price K are different.

The payoffs and prices of European calls and puts and gap calls can be expressed in terms of the asset and bond binaries.

For gap call options we have $Gc\left(x,t\right)=A_E^+\left(x,t\right)-KB_E^+\left(x,t\right)$

For a European call option we have $c\left(x,t\right)=A_K^+\left(x,t\right)-KB_K^+\left(x,t\right)$

For a European put option we have $p\left(x,t\right)=KB_K^-\left(x,t\right)-A_K^-\left(x,t\right)$

Definition:

A First order Q-binary option has payoff at time T defined by
$Q_E^s\left(X_T, T, K\right) = s\left(X_T - K\right)\mathbb{I}\left(sX_T > sE\right)$

This contract has value $Q_E^s\left(x,t,K\right) = s\left(A_E^s\left(x,t\right) - KB_E^s\left(x,t\right)\right)$
This is a generalization of the gap option

DUAL EXPIRY OPTIONS

second order asset binary option
defined over a single asset
matures at time T_2 and
payoff at that time given by

$$A_{E_1,E_2}^{s_1,s_2}\left(X_{T_2},T_2\right)=X_{T_2}\mathbb{I}\left(s_2X_{T_2}>s_2E_2\right)\mathbb{I}\left(s_1X_{T_1}>s_1E_1\right)$$

second order bond binary option,
defined over a single asset
matures at time T_2 and
has a payoff at that time given by

$$B_{E_1,E_2}^{s_1,s_2}\left(X_{T_2},T_2\right)=\mathbb{I}\left(s_2X_{T_2}>s_2E_2\right)\mathbb{I}\left(s_1X_{T_1}>s_1E_1\right)$$

These contracts have 2 exercise dates T_1, T_2 and 2 exercise prices E_1, E_2 .

The value of these contracts at time $t < T_1 < T_2$ are

$$A_{E_1,E_2}^{s_1,s_2}\left(x,t\right) = xe^{-q\tau_2} N_2\left(s_1 d_1^{'}, s_2 d_2^{'}; s_1 s_2 \rho\right)$$

$$B_{E_1,E_2}^{s_1,s_2}\left(x,t\right) = e^{-r\tau_2} N_2\left(s_1 d_1, s_2 d_2; s_1 s_2 \rho\right)$$

where $\rho = \sqrt{\tau_1/\tau_2}, \tau_2 = T_2 - t, \tau_1 = T_1 - t$

$$d_i = \frac{1}{\sigma\sqrt{\tau_i}}\left[\log\left(\frac{x}{E_i}\right) + \left(r - q - \frac{1}{2}\sigma^2\right)\tau_i\right], d_i^{'} = d_i + \sigma\sqrt{\tau_i}$$

$$N_2\left(x, y; \rho\right)$$

$$= \Pr\left(Z_1 < x, Z_2 < y \mid Z_i \sim N(0,1), corr\left(Z_1, Z_2\right) = \rho\right)$$

Dual expiry power options

We now consider options with a payoff at time T_2 of the form

$$f\left(X_1, X_2, T_1, T_2\right) = X_2{}^n \mathbb{I}\left(s_2 X_{T_2} > s_2 E_2\right) \mathbb{I}\left(s_1 X_{T_1} > s_1 E_1\right)$$

Where

$X_i = X_{T_i}$ is the stock price at time T_i and

$0 < t < T_1 < T_2$ and

$s_i \in \{-1, +1\}$ is the sign of the inequality in the indicator function for time i and

E_i is a positive constant and is the exercise price at the times T_i which defines the payoff.

This will be useful when we consider ESOs.
We denote the value of this by $P_{K_1,K_2}^{S_1,S_2}\left(x,t,n,T_1,T_2\right)$

and it is $e^{-r\left(T_2-t\right)}E\left\{X_{T_2}{}^{n}\mathbb{I}_{\left(S_1X_{T_1}>S_1K_1\right)}\mathbb{I}_{\left(S_2X_{T_2}>S_2K_2\right)}\mid X_t=x\right\}$

The value of this payoff at time $t<T_1$ is the function

$$P_{K_1K_2}^{S_1S_2}\left(x,t,n,T_1,T_2\right)$$

$$=x^n e^{\gamma(n)\tau_2}N_2\left(s_1\left(d_1+n\sigma\sqrt{\tau_1}\right),s_2\left(d_2+n\sigma\sqrt{\tau_2}\right),s_1s_2\rho\right)$$

$$where\ \tau_2=T_2-t,\ \tau_1=T_1-t,\rho=\sqrt{\frac{\tau_1}{\tau_2}},\ d_i=\frac{1}{\sigma\sqrt{\tau_i}}\left(\ln\frac{x}{K_i}+\left(r-q-\frac{1}{2}\sigma^2\right)\tau_i\right)$$

Valuation methods: PDE & Discounted Expectations

The main methods used for deriving analytic formulae for options in the black scholes framework are the PDE approach and the discounted expectations approach. For barrier options the PDE approach often works better and is easier to apply.

The Black Scholes Partial Differential Equation

It was shown by Black and Scholes (1973) that the economic value $V(x,t)$ of an option defined over a stock x as at time t satisfies the partial differential equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 X^2 \frac{\partial^2 V}{\partial X^2}+(r-q)X\frac{\partial V}{\partial X}-rV=0$$

It is convenient to write this in operator form $L\{V(x,t)\}=0$

The valuation function / formula for our option contract must be a solution of this PDE but it must also satisfy some conditions known as ***boundary conditions*** about the behavior of the function at maturity and possibly at other times.

Consider a call option with maturity *T*, and exercise price *K*. The boundary conditions satisfied by the European call option value are:

$$c(X,T) = (X - K)^+$$

$$c(0,t) = 0$$

$$c(X,t) \to Xe^{-q(T-t)} - Ke^{-r(T-t)} \text{ as X} \to \infty$$

Black Scholes Option Pricing formula

The Black Scholes pde for a European call option can be solved analytically. The solution for the call option is

$$c = X_0 e^{-qt} N(d_1) - Ke^{-rt} N(d_2)$$

where $d_{1,2} = \dfrac{1}{\sigma\sqrt{t}}\left[\ln(X_0/K) + \left(r - q \pm \dfrac{1}{2}\sigma^2\right)t\right]$

The stock price at any particular time has a lognormal distribution and

$$X_t \overset{d}{=} X_0 e^{\left(r-q-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}Z}, Z \sim N(0,1)$$

Under the risk neutral distribution we have

$$X_t > K \Leftrightarrow Z > -\frac{1}{\sigma\sqrt{t}}\left[\ln\left(X_0/K\right)+\left(r-q-\frac{1}{2}\sigma^2\right)t\right]$$

Expectation of the payoff at time t for a European call option is:

$$E(C)=\left[X_0 e^{(r-q)t}N(d_1)-KN(d_2)\right]$$

where $d_{1,2}=\frac{1}{\sigma\sqrt{t}}\left[\ln\left(X_0/K\right)+\left(r-q\pm\frac{1}{2}\sigma^2\right)t\right]$.

discount this at the risk free rate & we obtain the price of the option – the same as given by solving the PDE

Barrier options

Barrier options are a type of path dependent option. The payoff at maturity depends on whether or not the realised asset price path reaches a certain level called the barrier level. There are two main types of barrier option contracts.

A “knock out” option expires worthless if the stock price path crosses the barrier level before the maturity date. Otherwise, it has the payoff on some standard option contract at maturity.

A “knock in” option is one where the option expires worthless unless the stock price path crosses the barrier level before the maturity date. If it does cross the barrier level, then the option payoff will be that of some standard option contract (e.g. a put or a call etc). In this case the option is said to be “knocked in”.

This gives us 4 types of barrier options:
the down and out,
the down and in,
the up and out and
the up and in.

The “standard contract” that defines the payoff could be a call or a put. This gives a total of 8 different combinations of barrier option types, the “classical barrier options”.

However the standard contract could be something else than a call or put.

Extending the idea of barrier options

We can extend the concept of a barrier option in various ways, by having "standard payoffs" that differ from the European calls and puts, and by considering barrier options over other barrier options, i.e. sequential and compound barrier options.

The standard option has payoff at maturity $V_S\left(X_T,T\right)=f\left(X_T\right)$
e.g. the call option payoff is $f\left(X_T\right)=\max\left(X_T-K,0\right)=\left(X_T-K\right)\mathbb{I}\left(X_T>K\right)$.

Notation:

1) $V_S\left(X_T,T\right)=f\left(X_T\right)$ is the "standard option payoff at time T

2) $V_b^s\left(X_T,T\right)=f\left(X_T\right)\mathbb{I}\left(sX_T>sb\right), s=\pm 1$
are the (up, down) binary payoffs for barrier b

$$V_S\left(x,t\right)=e^{-r\tau}E\left\{V_S\left(X_T,T\right)\mid X_t=x\right\}$$

is the value of the standard payoff

$m_T=\min\left\{X_s:0\le s\le T\right\}$ is the minimum stock price over the time interval $\left(0,T\right)$

$M_T=\max\left\{X_s:0\le s\le T\right\}$ is the maximum stock price over the time interval $\left(0,T\right)$

A **down and out barrier option** is defined relative to some "barrier level" b

This option has a payoff at maturity T defined as

$$V_{do}\left(X_T, T\right) = f\left(X_T\right)\mathbb{I}\left(m_T > b\right)$$

Provided that at time 0, we have $X_0 > b$.
If not the contract would have zero value.

A **down and in barrier option** has a payoff at maturity T defined as

$$V_{di}\left(X_T, T\right) = f\left(X_T\right)\mathbb{I}\left(m_T < b\right)$$

Provided that at time 0, we have $X_0 > b$. If not the contract would have the same value as the standard contract.

An **up and in barrier option** has a payoff at maturity T defined as

$$V_{ui}\left(X_T,T\right)=f\left(X_T\right)\mathbb{I}\left(M_T>b\right)$$

Provided that at time 0, we have $X_0<b$. If not the contract would have the same value as the standard contract.

An **up and out barrier option** has a payoff at maturity T defined as

$$V_{uo}\left(X_T,T\right)=f\left(X_T\right)\mathbb{I}\left(M_T<b\right)$$

Provided that at time 0, we have $X_0<b$. If not the contract would have zero value.

Example: the down and out call option with exercise price K and barrier *b* has payoff

$$V_{do}\left(X_T,T\right)=\left(X_T-K\right)\mathbb{I}\left(X_T>K\right)\mathbb{I}\left(m_T>b\right)$$
$$=X_T\mathbb{I}\left(X_T>K\right)\mathbb{I}\left(m_T>b\right)-K\mathbb{I}\left(X_T>K\right)\mathbb{I}\left(m_T>b\right)$$

Barrier option PDEs

The **down and out** option $V_{DO}(x,t)$ will pay $f(x)$ at maturity provided it is not knocked out before expiry by the stock price falling to lower constant barrier level $x = b$ over the option lifetime.

It follows that $V_{DO}(x,t)$ satisfies the PDE / BVP

1. $L\{V_{DO}\} = 0$ on the domain
2. $V_{DO}(b,t) = 0, t < T$, $V_{DO}(x,T) = f(x)$

Similarly the other types of barrier option satisfy PDEs too.

The **down and in** option $V_{DI}\left(x,t\right)$ will expire worthless unless the lower constant barrier level $x=b$ is reached sometime before the options expiry.

It therefore satisfies the PDE / BVP

1. $L\left\{V_{DI}\right\}=0$ on the domain $D=\left\{\left(x,t\right):x>b,0<t<T\right\}$

   $V_{DI}\left(x,T\right)=0$, $V_{DI}\left(b,t\right)=V_S\left(b,t\right),t<T$

The other 2 barrier options also satisfy a PDE subject to certain boundary conditions

<u>The method of images for pricing barrier options</u>

Definition [Buchen (2001)]

Given any solution $V(x,t)$ of the BS PDE then
<u>the image</u> w.r.t. the barrier level $x = b$ is the function

$$\overset{*}{V}(x,t) = \left(\frac{b}{x}\right)^{\alpha} V\left(\frac{b^2}{x}, t\right) \text{ where } \alpha = 2\frac{(r-q)}{\sigma^2} - 1$$

Sometimes we use the notation $I_b\{f(x)\}$ for this

Theorem: [Method of Images Buchen (2001)] :

Given any payoff function $f\left(x\right)$, the solution to the down-and-out barrier problem is given by $V_{do}\left(x,t\right)=V_b\left(x,t\right)-\overset{*}{V}_b\left(x,t\right)$

$V_b\left(x,t\right)$ is the solution of another PDE problem:

$$L\left\{V_b\right\}=0$$

$$D=\left\{\left(x,t\right):x>0,0<t<T\right\},$$

$$V_b\left(x,T\right)=f\left(x\right)\mathbb{I}_{(x>b)}$$

The PDE for $V_b\left(x,t\right)$ has a simpler domain and boundary condition and is usually easier to solve. In fact it will often turn out to be a European type of option and have a nice closed form valuation formula.

Symmetry relations

for any given payoff function $f(x)$, it transpires that all four related barrier option prices follow given the price of one of the barrier option prices and the corresponding European option $V_S(x,T)$ as follows:

$$V_{DO}(x,t) = V_b^+(x,t) - I_b\left[V_b^+(x,t)\right]$$

$$V_{DI}(x,t) = V_S(x,t) - V_{DO}(x,t)$$

$$V_{UI}(x,t) = I_b\left[V_S(x,t)\right] + V_b^+(x,t) - I_b\left[V_b^+(x,t)\right]$$

$$V_{UO}(x,t) = \left(V_S(x,t) - I_b\left[V_S(x,t)\right]\right) - \left(V_b^+(x,t) - I_b\left[V_b^+(x,t)\right]\right)$$

In summary, these results allow us to express the price of any of the 4 types of barrier options

in terms of the function

$$V_b^s(x,t) = e^{-r(T-t)} E_Q\left\{ f(X_T) I(sX_T > sb) \mid X_t = x \right\}$$

and the standard option value

$$V_S(x,t) = e^{-r(T-t)} E_Q\left\{ f(X_T) \mid X_t = x \right\}$$

this holds true for any payoff function $f(X_T)$

example: the down and out call option

the standard contract with exercise price K has payoff

$$V_S(x,T) = (x-K)\mathbb{I}(x>K)$$

The "up binary" version of this contract has payoff

$$V_b^+(x,T) = (x-K)\mathbb{I}(x>K)\mathbb{I}(x>b)$$

We can write this as

$$V_b^+(x,T) = (x-K)\mathbb{I}(x>M)$$

Where M is the max of K and b

This is the payoff from a gap call option with exercise price M and strike price K

The value of the up binary contract is thus expressed in terms of asset and bond binary options

$$V_b^+(x,t) = A_M^+(x,t) - KB_M^+(x,t)$$

The value of the standard contract is

$$V_S(x,t) = A_K^+(x,t) - KB_K^+(x,t)$$

The value of the down and out barrier call option is

$$V_{do}(x,t) = V_b^+(x,t) - I_b\left[V_b^+(x,t)\right]$$

$$V_{do}(x,t) = V_b^+(x,t) - (b/x)^{\alpha} V_b^+\left(b^2/x,t\right)$$

Results about images

Given any solution $V(x,t)$ of the Black Scholes PDE, the image solution with respect to the barrier $x=b$ has several useful properties.

The image of the image of V is V $I_b\left\{I_b\left\{V(x,t)\right\}\right\}=V(x,t)$

If V is a solution of the PDE then so is its image $LV(x,t)=0 \Rightarrow L\overset{*}{V}(x,t)=0$

The function V and its image $\overset{*}{V}$ coincide at the barrier $x=b \Rightarrow V(x,t)=\overset{*}{V}(x,t)$

$$I_b\left\{\lambda V(x,t)+\mu U(x,t)\right\}=\lambda I_b\left\{V(x,t)\right\}+\mu\left\{I_b U(x,t)\right\}$$

$$I_b\left\{f(x)I(x>A)\right\}=I_b\left\{f(x)\right\}I\left(x<b^2/A\right)$$

The power option with term T, power n was considered earlier. The value of this contract at time $t < T$ is $P\left(x,t,n\right) = x^n e^{\gamma(n)\tau}$

The image with respect to the barrier b is

$$I_b\left\{P\left(x,t,n\right)\right\} = \left(b/x\right)^{\alpha}\left(b^2/x\right)^n e^{\gamma(n)\tau}$$
$$= b^{\alpha+2n} x^{-(n+\alpha)} e^{\gamma\left(-(n+\alpha)\right)\tau} = b^{\alpha+2n} P\left(x,t,-\left(n+\alpha\right)\right)$$

This is a multiple $b^{\alpha+2n}$ of another power option $P\left(x,t,-\left(n+\alpha\right)\right)$

The binary power option:

The binary power contract with term T, power n and exercise price K has payoff $P_K^s\left(X_T,T,n\right)=X_T{}^n I_{\left(sX_T>sK\right)}$

the value of this contract at time $t<T$ is

$$P_K^s\left(x,t,n,T\right)=x^n e^{\gamma(n)\tau}N\left(s\left(d\left(x,K,\tau\right)+n\sigma\sqrt{\tau}\right)\right)$$

where $d\left(x,K,\tau\right)=\dfrac{1}{\sigma\sqrt{\tau}}\left(\ln\dfrac{x}{K}+\left(r-q-\dfrac{1}{2}\sigma^2\right)\tau\right)$ and $\tau=T-t$

The image of this with respect to the barrier b is:

$$I_b\left\{P_K^s\left(x,t,n\right)\right\}=b^{2n+\alpha}P_{b^2/K}^{-s}\left(x,t,-\left(n+\alpha\right)\right)$$

We see that the image is a multiple of another binary power option with opposite sign, a different power $-\left(n+\alpha\right)$ and a different exercise price b^2/K

Special Cases:

1. When n=0 we have the bond binary contract

$$P_K^s\left(x,t,0\right)=e^{-r\tau}N\left(sd_2\left(x,K\right)\right)=B_K^s\left(x,t\right)$$

$$I_b\left\{P_K^s\left(x,t,0\right)\right\}=b^{\alpha}P_{b^2/K}^{-s}\left(x,t,-\alpha\right)$$

$$=b^{\alpha}x^{-\alpha}e^{-r\tau}N\left(-s\left(d_2\left(x,b^2/K\right)-\alpha\sigma\sqrt{\tau}\right)\right)$$

2.when n=1 we have the asset binary contract

$$P_K^s\left(x,t,1\right)=xe^{-t\tau}N\left(s\left(d_2\left(x,K\right)-\alpha\sigma\sqrt{\tau}\right)\right)=A_K^s\left(x,t\right)$$

The image of this is $I_b\left\{P_K^s\left(x,t,1\right)\right\}=b^{2+\alpha}P_{b^2/K}^{-s}\left(x,t,-\left(1+\alpha\right)\right)$

$$=b^{\alpha+2}e^{-q\tau}x^{-(\alpha+1)}N\left(-\left(d_2\left(x,b^2/K\right)-\left(1+\alpha\right)\sigma\sqrt{\tau}\right)\right)$$

3.when $n=\beta_1$ the positive root of $\gamma(n)=0$

value of contract is $P_K^s(x,t,\beta_1)=x^{\beta_1}N\left(s\left(d_2(x,K)+\beta_1\sigma\sqrt{\tau}\right)\right)$

$$I_b\left\{V_K^s(x,t,\beta_1)\right\}=b^{\beta_1-\beta_2}P_{b^2/K}^{-s}(x,t,\beta_2)=b^{\beta_1-\beta_2}x^{\beta_2}N\left(-s\left(d_2\left(x,b^2/K\right)+\beta_2\sigma\sqrt{\tau}\right)\right)$$

This is $b^{\beta_1-\beta_2}$ units of a power option with power β_2 and exercise price b^2/K and sign $-s$

Dual expiry power options

The value is a discounted conditional expectation

$$P_{K_1,K_2}^{S_1,S_2}\left(x,t,n,T_1,T_2\right)$$

$$= e^{-r\left(T_2-t\right)} E\left\{ X_{T_2}{}^{n} \mathbb{I}_{\left(S_1 X_{T_1} > S_1 K_1\right)} \mathbb{I}_{\left(S_2 X_{T_2} > S_2 K_2\right)} \mid X_t = x \right\}$$

The image of the function $P_{K_1,K_2}^{S_1,S_2}\left(x,t,n,T_1,T_2\right)$ is

$$I_b\left[P_{K_1,K_2}^{S_1,S_2}\left(x,t,n,T_1,T_2\right)\right] = b^{2n+\alpha} P_{b^2/K_1,b^2/K_2}^{-S_1,-S_2}\left(x,t,-\left(n+\alpha\right),T_1,T_2\right)$$

Formulae for the 4 barrier call options

The down and out call: applying Buchen's MOI we obtain

$$V_{do}\left(x,t\right)=V_b\left(x,t\right)-I_b\left\{V_b\left(x,t\right)\right\}$$

$$=P_M^+\left(x,t,1\right)-KP_M^+\left(x,t,0\right)-b^{2+\alpha}P_{b^2/M}^-\left(x,t,-\left(\alpha+1\right)\right)+Kb^{\alpha}P_{b^2/M}^-\left(x,t,-\alpha\right)$$

The down and in call: $V_{DI}\left(x,t\right)=V_S\left(x,t\right)-V_{DO}\left(x,t\right)$

$$=P_K^+\left(x,t,1\right)-KP_K^+\left(x,t,0\right)-P_M^+\left(x,t,1\right)+KP_M^+\left(x,t,0\right)$$
$$+b^{2+\alpha}P_{b^2/M}^-\left(x,t,-\left(\alpha+1\right)\right)-Kb^{\alpha}P_{b^2/M}^-\left(x,t,-\alpha\right)$$

The up and in call:

$$V_{UI}\left(x,t\right)=I_b\left[V_S\left(x,t\right)\right]+V_b^{\,+}\left(x,t\right)-I_b\left[V_b^{\,+}\left(x,t\right)\right]$$

$$=b^{2+\alpha}P^{-}_{b^2/K}\left(x,t,-\left(\alpha+1\right)\right)-Kb^{\alpha}P^{-}_{b^2/K}\left(x,t,-\alpha\right)$$

$$+P^{+}_{M}\left(x,t,1\right)-KP^{+}_{M}\left(x,t,0\right)$$

$$-b^{2+\alpha}P^{-}_{b^2/M}\left(x,t,-\left(\alpha+1\right)\right)+Kb^{\alpha}P^{-}_{b^2/M}\left(x,t,-\alpha\right)$$

The up and out call:

$$V_{UO}\left(x,t\right)=V_S\left(x,t\right)-I_b\left\{V_S\left(x,t\right)\right\}-V_b\left(x,t\right)+I_b\left\{V_b\left(x,t\right)\right\}$$

$$=P_K^+\left(x,t,1\right)-KP_K^+\left(x,t,0\right)$$
$$-b^{2+\alpha}P_{b^2/K}^-\left(x,t,-\left(\alpha+1\right)\right)+Kb^{\alpha}P_{b^2/K}^-\left(x,t,-\alpha\right)$$
$$-P_M^+\left(x,t,1\right)+KP_M^+\left(x,t,0\right)$$
$$+b^{2+\alpha}P_{b^2/M}^-\left(x,t,-\left(\alpha+1\right)\right)-Kb^{\alpha}P_{b^2/M}^-\left(x,t,-\alpha\right)$$

<u>More building blocks</u>

Rebates

Some barrier option contracts pay a “rebate” to the holder of the option if either the contract payoff gets “knocked out”. The rebate is a fixed sum of money R paid at the time the option gets knocked out.

These rebates may be part of the structure of some ESOs, for instance as a cash bonus paid if and when the stock price reaches some specified level.

This rebate can be thought of and valued as a down and in (or an up and in respectively) barrier power option, with power β_1, the positive root of the equation $\gamma(n) = 0$ and with exercise price b.

Let the "standard" option have the payoff. $f\left(x\right)=R\left(\frac{x}{b}\right)^{\beta_1}$ for a power option with power $n=\beta_1$, the positive root of $\gamma\left(n\right)=0$.

The value of the standard contract at time t is

$$V_S\left(x,t\right)=\frac{R}{b^{\beta_1}}\left(x\right)^{\beta_1}e^{\gamma(\beta_1)\tau}=R\left(\frac{x}{b}\right)^{\beta_1}$$

We shall examine the up and in barrier version of this option, with barrier at $x=b$.

The up and in version of this option with barrier *b* pays a fixed amount R if and when the stock price crosses the barrier from below prior to maturity. This contract has value given by the expression $V_{ui}\left(x,t\right)=I_b\left\{V_b^-\left(x,t\right)\right\}+V_b^+\left(x,t\right)$

The value of this contract is

$$V_{UI}\left(x,t\right)=\frac{R}{b^{\beta_1}}\left[I_b\left\{P_b^{-}\left(x,t,\beta_1\right)\right\}+P_b^{+}\left(x,t,\beta_1\right)\right]$$

$$=\frac{R}{b^{\beta_1}}x^{\beta_1}N\left(d_2\left(x,K\right)+\beta_1\sigma\sqrt{\tau}\right)$$

$$+\frac{R}{b^{\beta_1}}b^{\beta_1-\beta_2}x^{\beta_2}N\left(\left(d_2\left(x,b^2/K\right)+\beta_2\sigma\sqrt{\tau}\right)\right)$$

Partial time up and in barrier power option

In this section we consider expressions for partial time barrier options for which the barrier monitoring window is a subset of the option's lifetime.

Consider a situation where at time T_1 we have the right to receive an up and in barrier power option with maturity T_2 and barrier b provided the stock price at time T_1 is below the level b.

The barrier is monitored over the time window (T_1, T_2) not over the entire time till maturity (t, T_2).

We want to value this right. The value of the contract at time t is

$$value = e^{-r\tau_1} E_Q \left\{ V_{UI}\left(X_{T_1}, T_1, T_2\right) \mathbb{I}\left(X_{T_1} < b\right) \mid X_t = x \right\}$$

A valuation formula for this contract is

$$value = P_{b,b}^{-1,+1}\left(x,t,n,T_1,T_2\right) + I_b\left[P_{b,b}^{+1,-1}\left(x,t,n,T_1,T_2\right)\right]$$

$$P_{K_1K_2}^{S_1S_2}\left(x,t,n,T_1,T_2\right)$$

$$= x^n e^{\gamma(n)\tau_2} N_2\left(s_1\left(d_{12} + n\sigma\sqrt{\tau_1}\right), s_2\left(d_{22} + n\sigma\sqrt{\tau_2}\right), s_1 s_2 \rho\right)$$

where

$$\tau_2 = T_2 - t,\ \tau_1 = T_1 - t, \rho = \sqrt{\frac{\tau_1}{\tau_2}},\ d_{i2} = \frac{1}{\sigma\sqrt{\tau_i}}\left(\ln\frac{x}{K_i} + \left(r - q - \frac{1}{2}\sigma^2\right)\tau_i\right)$$

Partial time up and out barrier call option

At time T_1 we have the right to receive an up and out barrier call option with maturity T_2 and barrier b and exercise price K, provided the stock price at time T_1 is below the level b.

The barrier is monitored over the time window (T_1, T_2)not over the interval (t, T_2). We want to value this right.

The value of the contract is

$$value = e^{-r\tau_1} E_Q \left\{ C_{UO}\left(X_{T_1}, T_1, T_2\right) \mathbb{I}\left(X_{T_1} < b\right) \mid X_t = x \right\}$$

The value can be written in terms of dual expiry power options and their images as follows:

$$
\begin{aligned}
&P_{b,K}^{-,+}\left(x,t,1,T_1,T_2\right)-KP_{b,K}^{-,+}\left(x,t,0,T_1,T_2\right)\\
&-\left(P_{b,M}^{-,+}\left(x,t,1,T_1,T_2\right)-KP_{b,M}^{-,+}\left(x,t,0,T_1,T_2\right)\right)\\
&-I_b\left[P_{b,K}^{+,+}\left(x,t,1,T_1,T_2\right)-KP_{b,K}^{+,+}\left(x,t,0,T_1,T_2\right)\right]\\
&+I_b\left[\left(P_{b,M}^{+,+}\left(x,t,1,T_1,T_2\right)-KP_{b,M}^{+,+}\left(x,t,0,T_1,T_2\right)\right)\right]
\end{aligned}
$$

Valuation of ESOs via exercise multiple approach

Hull and White (2004), claim that empirical studies indicate ESO's are exercised early as soon as the stock price reaches some multiple of the exercise price, or exercised at the final maturity date if that multiple is never reached.

For instance the multiple might be 120% of the exercise price. Their paper proposes a numerical valuation using the binomial for valuing the ESO using this assumption about the early exercise behavior. We next show how this approach to valuation of an ESOs allows us to value the contract as a combination of non standard barrier options, then we demonstrate how to value the ESO analytically. To motivate the discussion we consider the following hypothetical example.

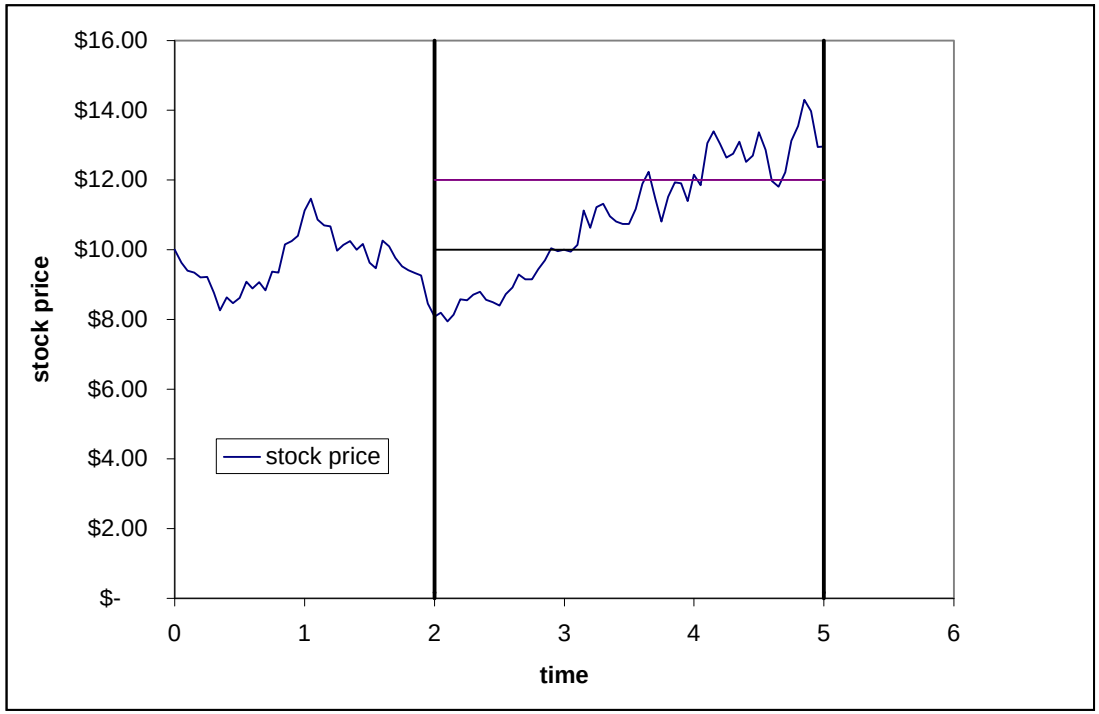
Hypothetical ESO: with

- a vesting period of $T_1 = 2$ years
- term to maturity of $T_2 = 5$ years
- initial stock price $X_0 = 10$
- exercise price of option $K = X_0$

The option can't be exercised during the vesting period. After the end of the vesting period the option can be exercised early. If exercised at time t where $T_1 < t < T_2$ then the payoff is a standard call option $payoff = \max\left(X_t - K, 0\right)$. The time interval $\left(T_1, T_2\right)$ is the "early exercise window".

We allow for early exercise by assuming that the option is exercised as soon as it reaches the specified level. This level is effectively a barrier for an "up and in barrier option". The option is a type of barrier option. However it is a "partial time" barrier option as it can't be exercised until after the vesting period is over.

Consider the diagram below.



The option can be exercised between time 2 and time 5. The initial stock price is \$10.00 and this is the same as the exercise price of the option. We assume that the option is exercised as soon as the stock price reaches 120% of the exercise price. For this example this event occurs at time 3.65 years.

Let $K = X_0$ be the exercise price of the ESO.
It is an at the money option at the time it is granted.

Let $B = \gamma K$ be the "barrier" level such that early exercise of the option will happen immediately if the stock price crosses the barrier from below during the early exercise window.

Let $\quad M = \max\left\{X_t : T_1 \le t \le T_2\right\}$

This is the maximum stock price level during the early exercise window $\left(T_1, T_2\right)$

Let $\quad s = \min\left\{T_1 \le t \le T_2 : X_t = B\right\}$

This is the first time during the early exercise window that the stock price reaches the barrier level, if such event occurs.

We consider 3 different scenarios. These are mutually exclusive and exhaustive.

<u>Case 1</u>: $X_{T_1} > B$ This means the ESO is exercised early at time T_1 for a payoff of $\max\left(X_{T_1} - K, 0\right)$ because the stock price has reached the level at which the executive will exercise and will do so immediately.

<u>Case 2:</u> $X_{T_1} < B$ and $M > B$: This means the ESO was not exercised at the beginning of the early exercise window but reached the critical stock price before the maturity date. The ESO is exercised early at time s for a payoff of $\max\left(X_s - K, 0\right) = \max\left(B - K, 0\right) = \left(\gamma - 1\right) S_0 = R$

<u>Case 3:</u> $X_{T_1} < B$ and $M < B$: This means the conditions for earlier exercise of the ESO were not met so the ESO is exercised at maturity (time T_2) for a payoff of $\max\left(X_{T_2} - K, 0\right)$, provided of course that it is in the money at that time.

The payoff structure

We can write the payoff as a combination of 3 different payoffs $P = P_1 + P_2 + P_3$. These payoffs are mutually exclusive and exhaustive. The amount and timing of the payoffs are as follows:

$$P_1 = \max\left(X_{T_1} - K, 0\right)\mathbb{I}_{\left(X_{T_1} > B\right)} \qquad \text{paid at time } T_1$$

$$P_2 = \mathbb{I}_{\left(X_{T_1} < B\right)}\mathbb{I}_{(M > B)}R \qquad \text{paid at time } s$$

$$P_3 = \mathbb{I}_{\left(X_{T_1} < B\right)}\mathbb{I}_{(M < B)}\max\left(X_{T_2} - K, 0\right) \qquad \text{paid at time } T_2$$

The first payoff is the payoff from a gap call option that matures at time T_1.

The second payoff is the payoff from an "up and in barrier power option with power β_1", which is the same thing as the **<u>rebate</u>** from an up and out call option. If the barrier level is reached we receive a fixed amount of cash. However it is a **partial time** up and in barrier power option.

The third payoff is the payoff from an up and out barrier call option, but it is a "partial time up and out barrier call call option".

Valuing the first component

It is straightforward to value the first payoff at time $t < T_1$. The value of this component at time t before the vesting date is

$$value(P1) = xe^{-q\tau} N\left(d_1^{'}\right) - Ke^{-r\tau} N\left(d_2^{'}\right)$$

where

$$d_1^{'}, d_2^{'} = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{x}{M}\right) + \left(r - q \pm \frac{1}{2}\sigma^2\right)\tau\right],\ \tau = T_1 - t \text{ and } M = \max(K, B)$$

Valuing the second component:

Let's consider the 2nd component of the payoff and consider its valuation as at time T_1

At that time, the value of the payoff is the value of an up and in barrier power option provided the condition $\mathbb{I}\left(X_{T_1} < B\right) = 1$ holds.

The value of the payoff at time $t < T_1$ is the value of a partial time up and in barrier power option. A closed form formula for the value of this payoff at time $t < T_1$ gives the value as a combination of dual expiry power binary options.

The value is:

$$value(P2)$$

$$= \frac{R}{B^{\beta_1}} x^{\beta_1} N_2\left(-\left(d_{12} + \beta_1 \sigma \sqrt{\tau_1}\right), d_{22} + \beta_1 \sigma \sqrt{\tau_2}, -\rho\right)$$

$$+ \frac{R}{B^{\beta_2}} x^{\beta_2} N_2\left(-\left(d_{12} + \beta_2 \sigma \sqrt{\tau_1}\right), d_{22} + \beta_2 \sigma \sqrt{\tau_2}, -\rho\right)$$

where $\tau_2 = T_2 - t$, $\tau_1 = T_1 - t$, $\rho = \sqrt{\frac{\tau_1}{\tau_2}}$, $d_{i2} = \frac{1}{\sigma\sqrt{\tau_i}}\left(\ln\frac{x}{B} + \left(r - q - \frac{1}{2}\sigma^2\right)\tau_i\right)$

Valuing the third component

Let's consider the 3rd component of the payoff and consider its valuation as at time T_1. This is a partial time up and out barrier option with barrier B and exercise price K. The value of this payoff is

$$
\begin{aligned}
&value(P3) = V_1 - V_2 = \\
&P_{B,K}^{-,+}(x,t,1,T_1,T_2) - KP_{B,K}^{-,+}(x,t,0,T_1,T_2) \\
&-\left(P_{B,M}^{-,+}(x,t,1,T_1,T_2) - KP_{B,M}^{-,+}(x,t,0,T_1,T_2)\right) \\
&-I_B\left[P_{B,K}^{+,+}(x,t,1,T_1,T_2) - KP_{B,K}^{+,+}(x,t,0,T_1,T_2)\right] \\
&+I_B\left[\left(P_{B,M}^{+,+}(x,t,1,T_1,T_2) - KP_{B,M}^{+,+}(x,t,0,T_1,T_2)\right)\right]
\end{aligned}
$$

The image of a dual expiry power option can be obtained via the result

$$I_B\left[P_{K_1,K_2}^{s_1,s_2}\left(x,t,n,T_1,T_2\right)\right]=B^{2n+\alpha}P_{B^2/K_1,B^2/K_2}^{-s_1,-s_2}\left(x,t,-\left(n+\alpha\right),T_1,T_2\right)$$

The value of the ESO is the sum of the three components.

This value is

$$value=value\left(P1\right)+value\left(P2\right)+value\left(P3\right)$$

where the three terms on the right hand side are as shown above.

Conclusions:

We have covered the development of valuation formulae for power options, dual expiry power options and for various types of barrier option. We used the newly developed method of images for doing this.

Using these methods we derive an analytic formula for the pricing of an ESO using the Hull White exercise multiple approach, which involves modeling early exercise behavior by assuming exercise happens early if the stock price exceeds some multiple of the exercise price. This leads to the ESO being a combination of a gap option and two partial time barrier options. The valuation formulae involve the bivariate normal cdf.