australian
actuarial journal

2006
VOLUME 12
issue 2

Institute of Actuaries of Australia

australian
actuarial journal

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A Multiple State Model for Pricing and Reserving Private Long Term Care Insurance Contracts in Australia

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We seek to develop a model for pricing LTC insurance contracts in Australia using the disability prevalence rates contained in the 1998 Australian Bureau of Statistics (ABS) Survey of Disability, Ageing and Carers. We perform premium and reserve calculations by applying generalisations of Thiele's differential equation for a multiple state model within a Markov framework. Several sets of results are presented that both capture a varying range of possible scenarios and demonstrate the flexibility of the model.

The author is grateful to Professor David Dickson and two anonymous referees whose comments materially improved the exposition of this paper. Any errors or omissions, however, remain the responsibility of the author.

Keyword: *Long term care; Disability; Multiple State Model; Private Insurance; Markov Process; Thiele's Differential Equation.*

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1.0 Introduction

The long term care (LTC) system in Australia is characterised by an absence of risk pooling or a sophisticated user pays mechanism. The system, therefore, stands somewhat isolated from many of its counterparts overseas which combine private funding mechanisms such as private LTC insurance with their respective State and publicly funded welfare programs. With the exception of a limited number of accident compensation policies where LTC is insured if attributable to accident, Australian insurers do not currently engage in any form of LTC insurance business.

The primary objective of this paper, therefore, is to develop and test a multiple state model for pricing and reserving LTC insurance using currently available Australian data. In Leung (2004), a discrete time multiple state model was developed for projecting the needs and costs of LTC in Australia. In this paper, we relax the assumption of discrete time and model the underlying process in a continuous time Markov framework. The purpose of this is to enable calculation of transition intensities for application in Thiele's differential equation for pricing and reserving. The modelling framework and results presented in this paper may be used as a starting point for the development of LTC policies in Australia.

In this paper, we survey the relevant data currently available in Australia for pricing and reserving for LTC insurance, and follow this with a brief review of the existing LTC pricing and reserving literature emerging from Australia and abroad. Next, we develop the multiple state model and discuss the probabilistic structure used to calculate premiums and reserves for a set of illustrative hypothetical LTC insurance products. Finally, we analyse the sensitivities of the model and present further avenues for research.

2.0 Data Review

A number of data sets suitable for LTC analysis (typically hostel and nursing home data) are administered in Australia by the Department of Health and Ageing, the Australian Bureau of Statistics (ABS) and other government bodies. However, few are suitable for the modelling methodology undertaken here. In this section, we outline the basic data requirements for our methodology and our justification in selection of the ABS prevalence rate data (ABS 1998)

2.1 Data Requirements

Ideal data for LTC insurance pricing is a longitudinal data set that tracks both levels of disability and LTC utilisation patterns of a large representative population. As discussed by Meiners (1989), the benefit of longitudinal data for LTC pricing is primarily to enable an understanding of LTC utilisation changes as the cohort ages.

Many nations, including Australia, lack a systematic LTC data-reporting program enabling comprehensive information to be collected across service sectors, care programs and jurisdictions (Reif 1985). Given that Australia currently has no private insurance coverage for LTC, there is clearly a need to gather data on virtually all aspects of LTC cover including costs, risk management, marketing and underwriting. From a pure actuarial pricing and reserving perspective, utilisation/demand data for LTC segregated by age and sex in conjunction with changes to utilisation /demand (ie. functional changes) as a function of age are essential. The following sections discuss and evaluate the various options for obtaining this information.

2.2 Australian Bureau of Statistics Surveys (1981, 1988, 1993, 1998)

The ABS has published results of a number of surveys detailing Australian population data on persons with disabilities, older persons and persons who provide assistance to others due to their disabilities. The surveys are:

- Survey of Handicapped Persons (1981)
- Survey of Disabled and Aged Persons (1988)
- Survey of Disability, Ageing and Carers (1993)
- Survey of Disability, Ageing and Carers (1998)

These surveys provide the only comprehensive source of data concerning the functional capacities of persons in Australia on a population scale. The results from the Survey of Disability, Ageing and Carers conducted from 16 March to 29 May 1998 represent the most current information as at the time of writing. The data contained within those surveys and used for this paper are those that relate to core activity restrictions as detailed in Leung (2004).

Note that although the survey results are not categorised according to an activities of daily living (ADL) scale, the categorisation of data according to differing levels of core activity restriction (ie. differing levels of severity of disability) renders the results useful for the purposes of LTC pricing – and, furthermore, easily translatable to an ADL system. For instance, a claim for LTC may be allowed upon the failure of between 3 and 6 ADLs. Given that only persons who have either a severe or profound core activity restriction, by definition, require LTC, one could infer a severe core activity restriction being equivalent to the failure of 3 or 4 ADLs and a profound core activity restriction being equivalent to the failure of 5 or 6 ADLs.

As outlined, ideal data required for pricing and reserving LTC insurance contracts includes both the number of persons requiring LTC and the change in this demand as a cohort of persons ages.

Although non-longitudinal, the ABS survey data may conceivably be used to ascertain this information in a number of ways which will be discussed in Section 4.

We re-iterate at this stage that the data for pricing and reserving LTC insurance in Australia is far from ideal – restricting us largely to the prevalence rates contained in the 1998 ABS survey of Disability, Ageing and Carers. These data limitations inevitably influence many of the assumptions concerning methodology in this paper. We have tried to be as realistic as the data allows.

3.0 Literature Review

A range of methodologies may be applied to pricing LTC insurance including inception annuity approaches (Gatenby 1991) or risk renewal approaches (Beekman 1989). The chosen methodology in this paper is a multiple state modelling approach within a continuous time Markov framework with premiums and reserves calculated by means of applying generalisations of Thiele's differential equations. For brevity, we will refer to these as Thiele's differential equation for the remainder of the paper. This choice is motivated by the benefits of multiple state modelling being an accurate representation of the underlying insurance process, a greater degree of flexibility and scope for scenario testing and the ease of monitoring actual experience against expected at a practical level (Gatenby and Ward 1994, Robinson 1996 and Society of Actuaries Long-Term Care Insurance Valuation Methods Task Force 1995).

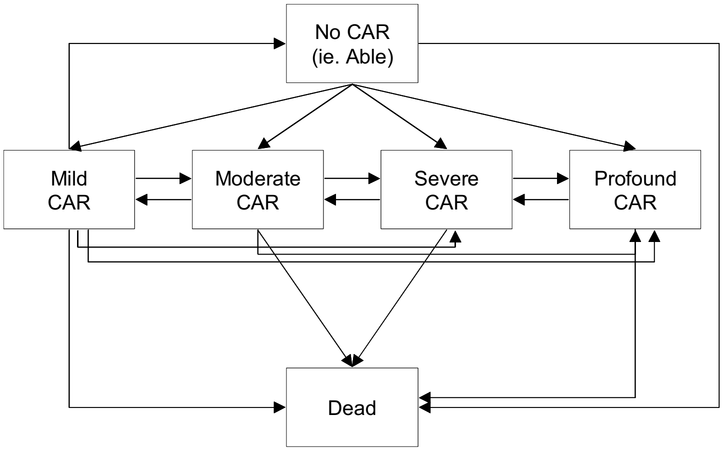
Multiple state models are prevalent in the actuarial literature in areas including life insurance (Pitacco 1995), permanent health insurance (PHI) in the UK (Waters 1984, Sansom and Waters 1988, Haberman 1993, Renshaw and Haberman 1998, Cordeiro 2001) and disability income insurance (Haberman and Pitacco 1999). It is therefore unsurprising that the suitability of multiple state modelling for LTC insurance has been well recognised and consequently applied. For instance, Levikson and Mizrahi (1994) consider an 'upper triangular' multiple state model in the general Markovian framework where three care levels are considered and the insured life proceeds through the deteriorating stages of ADL failure until death. Premium calculation is subsequently performed via a representation of the discounted value of future benefits in a particular care level as a random variable. Similar frameworks have been studied by Alegre et al (2002), who also consider a LTC system with no recoveries and premium calculations derived by calculating annuity values in discrete time for a life in a LTC claiming state. Moreover, the valuation of LTC annuities to price LTC insurance in continuous time has been discussed by Pitacco (1993) and Czado and Rudolph (2002).

Despite the wide range of methodologies considered abroad, only limited literature concerning pricing LTC insurance contracts in Australia has been published. The earliest paper, by Walker (1990), provides a brief introduction to the issues surrounding LTC insurance pricing and provides specimen net single and annual renewable premiums for a LTC benefit using illustrative morbidity rates for males, females and couples. Walsh and De Ravin (1995) perform similar calculations based on data sourced from the 1993 ABS survey of Disability, Ageing and Carers and calculated premium rates directly from prevalence rate data. The mathematical methodologies are not detailed in their respective papers, but it is clear that in both papers, calculations are based on an inception-annuity approach framework.

4.0 Model Specification And Defining Assumptions

In Leung (2004), we used a discrete time multiple state model as depicted in Figure 1. Here, we relax the assumption of discrete time and apply it in a continuous time framework. The motivation for this is to enable the calculation of transition intensities for the purpose of actuarial application – namely pricing and reserving using Thiele's differential equations. Note that we could have persisted with a discrete time process to price LTC cover using annuity functions as in Alegre et al (2002). However, it was felt that the greater practicality, flexibility and realism offered by using a Thiele's differential equation framework for pricing and reserving was a better route.

Figure 1: Transitions in the multiple state model.



Note that the model does not include an absorbing lapse state. While the inclusion of an additional absorbing state to account for lapses is preferable for insurance pricing purposes, its omission is solely attributable to unavailability of suitable data.

5.0 Estimating Transition Probabilities

Ideally, we would like to estimate transition intensities directly from our data. However, the 1998 ABS survey data is in the form of prevalence rates at one snapshot of time. We thus have no information as to when transitions to the various core activity restriction categories occur. We outline possible approaches to this problem in the following.

5.1 Maximum Likelihood Estimation

One possibility may be to compare prevalence rates over two or more consecutive ABS surveys (for instance 1993 and 1998) and calculate maximum likelihood estimates of the *t*-year probability

${}_t p_x^{ab}$ of a life aged (x) making a transition from state *a* to state *b* using an equation of the form:

$$ {}_t p_x^{ab} = \frac{n_{x,x+t}^{ab}}{\sum\limits_j n_{x,x+t}^{aj}} \tag{1} $$

where $n_{x,x+t}^{ab}$ is the number of persons in state *a,* aged *x* in 1993, say, and in state *b*, aged *x+5* in 1998. This type of approach has been undertaken using US National Long Term Care Survey (NLTCS) data by several studies including Manton (1988) and Manton et al (1993).

We have a number of reservations about implementing such an approach using ABS survey data. Firstly, the ABS survey data is not longitudinal. That is, persons have not been individually tracked as is the case with the NLTCS surveys. Secondly, survey design changes over consecutive surveys will inevitably render any calculated transition probabilities inaccurate. Madden and Wen (2001) argue that an increase in prevalence from 1993 to 1998 does not reflect a substantial increase in underlying disability but rather a change in disability survey design. A similar view is put forward by Davis et al (2001) who suggest that over half of the increase in prevalence between 1993 and 1998 is due to changes in survey method. This approach was thus not favoured in this paper.

5.2 Approximation from 1-step Transition Probabilities

We therefore use 1-year transition probabilities as calculated in Leung (2004) to estimate a set of transition intensities. A detailed discussion of estimating the 1-step transition probabilities in discrete time and the associated parameters is given in Leung (2004).

The 1-step transition probabilities at 10-yearly age intervals are reported in Tables 1 and 2 and illustrated in Figures 2 and 3 for males and females (from the able state) respectively.

Several observations should be made at this point.

Transition probabilities for both males and females generally behave as expected with transition probabilities to disability states increasing with age.

1. Transition through disability levels is reasonably progressive. That is, given that a transition out of the disabled state occurs, there is a higher probability of moving to a lower disability level than directly to a more severe disability level. At higher ages, however, transition to the profound core activity restriction state appears to mildly exceed other intermediate disability levels. This seems reasonable owing to the effects of ageing and chronic frailty.

2. Transition probabilities out of the disabled state appear higher for males than females.

3. Given that a transition out of the disabled state occurs, transition to profound or severe core activity restriction states appears higher for females than males.

4. Mortality in the profound or severe core activity restriction states is higher for males than females.

5. Points 4 and 5 above are particularly interesting as they form the basis of an *a priori* expectation that the likelihood of LTC utilisation by females will be higher than by males in the Australian population, therefore resulting in more expensive premiums for females.

Table 1: Male 1-step transition probabilities

	Able	Mild	Moderate	Severe	Profound	Dead
Able						
20	0.990045	0.005229	0.001793	0.000926	0.000808	0.001199
30	0.988251	0.006233	0.002137	0.001104	0.000963	0.001313
40	0.983648	0.008726	0.002992	0.001545	0.001348	0.001742
50	0.971556	0.014860	0.005095	0.002632	0.002295	0.003562
60	0.940283	0.029544	0.010130	0.005234	0.004564	0.010245
70	0.897377	0.043650	0.015034	0.007801	0.006833	0.029305
80	0.702715	0.119224	0.046477	0.027297	0.027065	0.077222
Mild						
20	0.15	0.844587	0.002142	0.001107	0.000965	0.001199
30	0.15	0.843664	0.002554	0.001319	0.001150	0.001313
40	0.15	0.841226	0.003575	0.001847	0.001610	0.001742
50	0.15	0.834462	0.006088	0.003145	0.002742	0.003562
60	0.15	0.815941	0.012106	0.006254	0.005454	0.010245
70	0.15	0.785242	0.017965	0.009322	0.008166	0.029305
80	0.15	0.652276	0.055540	0.032620	0.032342	0.077222
Moderate						
20	0	0.15	0.846325	0.001322	0.001153	0.001199
30	0	0.15	0.845736	0.001576	0.001375	0.001313
40	0	0.15	0.844127	0.002207	0.001924	0.001742
50	0	0.15	0.839403	0.003758	0.003277	0.003562
60	0	0.15	0.825764	0.007474	0.006518	0.010245
70	0	0.15	0.799797	0.011140	0.009758	0.029305
80	0	0.15	0.695148	0.038981	0.038649	0.077222
Severe						
20	0	0	0.1	0.895797	0.001378	0.002825
30	0	0	0.1	0.893162	0.001643	0.005195
40	0	0	0.1	0.887611	0.002300	0.010090
50	0	0	0.1	0.877522	0.003916	0.018562
60	0	0	0.1	0.860314	0.007789	0.031897
70	0	0	0.1	0.832916	0.011661	0.055423
80	0	0	0.1	0.748219	0.046185	0.105596
Profound						
20	0	0	0	0.05	0.945549	0.004451
30	0	0	0	0.05	0.940923	0.009077
40	0	0	0	0.05	0.931562	0.018438
50	0	0	0	0.05	0.916438	0.033562
60	0	0	0	0.05	0.896451	0.053549
70	0	0	0	0.05	0.868459	0.081541
80	0	0	0	0.05	0.816030	0.133970
Dead						
20	0	0	0	0	0	1
30	0	0	0	0	0	1
40	0	0	0	0	0	1
50	0	0	0	0	0	1
60	0	0	0	0	0	1
70	0	0	0	0	0	1
80	0	0	0	0	0	1

Table 2: Female 1-step transition probabilities

	Able	Mild	Moderate	Severe	Profound	Dead
Able						
20	0.991502	0.004906	0.001299	0.0009463	0.000929	0.000417
30	0.990264	0.005612	0.001486	0.0010826	0.001063	0.000492
40	0.986776	0.007486	0.001983	0.0014443	0.001418	0.000893
50	0.977310	0.012414	0.003290	0.0023981	0.002357	0.002231
60	0.952880	0.024985	0.006659	0.0048804	0.004823	0.005772
70	0.920111	0.037427	0.010481	0.0080711	0.008381	0.015529
80	0.745522	0.088349	0.033458	0.0348423	0.048925	0.048903
Mild						
20	0.15	0.845614	0.001624	0.0011830	0.001162	0.000417
30	0.15	0.844967	0.001858	0.0013533	0.001329	0.000492
40	0.15	0.843050	0.002479	0.0018054	0.001773	0.000893
50	0.15	0.837712	0.004113	0.0029977	0.002946	0.002231
60	0.15	0.823774	0.008324	0.0061008	0.006029	0.005772
70	0.15	0.800804	0.013102	0.0100893	0.010476	0.015529
80	0.15	0.654558	0.041825	0.0435550	0.061159	0.048903
Moderate						
20	0	0.15	0.846652	0.0014788	0.001452	0.000417
30	0	0.15	0.846155	0.0016917	0.001661	0.000492
40	0	0.15	0.844634	0.0022569	0.002217	0.000893
50	0	0.15	0.840339	0.0037473	0.003683	0.002231
60	0	0.15	0.829065	0.0076264	0.007537	0.005772
70	0	0.15	0.808763	0.0126122	0.013096	0.015529
80	0	0.15	0.670198	0.0544463	0.076453	0.048903
Severe						
20	0	0	0.1	0.8961415	0.001815	0.002043
30	0	0	0.1	0.8935488	0.002077	0.004374
40	0	0	0.1	0.8879883	0.002771	0.009241
50	0	0	0.1	0.8781655	0.004604	0.017231
60	0	0	0.1	0.8631546	0.009421	0.027424
70	0	0	0.1	0.8419825	0.016371	0.041647
80	0	0	0.1	0.7271528	0.095570	0.077277
Profound						
20	0	0	0	0.05	0.946331	0.003669
30	0	0	0	0.05	0.941744	0.008256
40	0	0	0	0.05	0.932411	0.017589
50	0	0	0	0.05	0.917769	0.032231
60	0	0	0	0.05	0.900924	0.049076
70	0	0	0	0.05	0.882235	0.067765
80	0	0	0	0.05	0.844349	0.105651
Dead						
20	0	0	0	0	0	1
30	0	0	0	0	0	1
40	0	0	0	0	0	1
50	0	0	0	0	0	1
60	0	0	0	0	0	1
70	0	0	0	0	0	1
80	0	0	0	0	0	1

Figure 2: Male 1-step transition probabilities

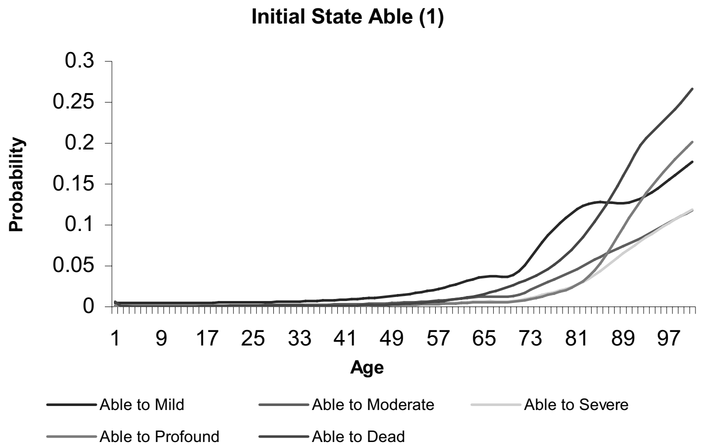
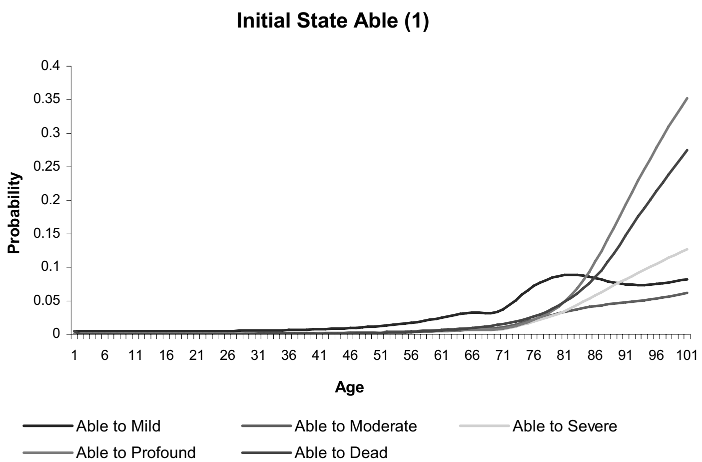


Figure 3: Female 1-step transition probabilities



6.0 Estimation of Transition Intensities

Given our inability to estimate transition intensities directly from our data, we calculate transition intensities in the model using estimated transition probabilities. Note, however, that since we have approximated 1-year transition probabilities using 1-step transition probabilities from Rickayzen and Walsh's (2002) framework, this creates difficulties when transforming transition probabilities to transition intensities. This is because the discrete time framework proposed by Rickayzen and Walsh (2002) is characterised by incomplete communication of all states. This inevitably leads to structural inconsistencies in our transition intensity matrix, with several off-diagonal entries estimated as negative values. One possible way of approaching this difficulty is to re-estimate a set of transition probabilities inclusive of a full set of recovery transitions – consistent with a Markov framework. Given our data limitations, however, this was not possible. We chose to pursue estimating transition intensities from our existing 1-step transition probabilities and couple this with a constraining algorithm to force the negative transition intensities to lie in the feasible region. Our justification for this is twofold. First, even if we were able to estimate accurately a full set of transition probabilities including all recovery transitions, this does not guarantee that the problem of negative transition intensities would occur (for example, Pritchard (2002)). Second, the recovery transitions that are absent under the current framework are likely to be small. For the ultimate purpose of this paper which is to price and reserve, the impact of this inconsistency is minimal and in any case, should be encapsulated within the bounds of our sensitivity analysis.

6.1 Calculating Transition Intensities from Transition Probabilities

We impose a Markov assumption to describe the process in our model. That is, we consider a stochastic process $\{S(t),\ 0 < t < \infty\}$ with state space {1,2,…,6} where *S(t)* represents the state of the process at time *t.* $\{S(t),\ 0<t<\infty\}$ is a continuous time Markov chain if for states $g,h \in \{1,2,...6\}$ and $x,\ t \geq 0$,

$$\Pr\{S(x+t)=h \mid S(x)=g, S(r) \text{ for } 0 \leq r \leq x\} = \Pr\{S(x+t)=h \mid S(x)=g\} \quad (2)$$

In other words, the future development of *S(t)* can be determined only from its present state and without regard to the process history. We denote ${}_t p_x^{gh} = \Pr\{S(x+t)=h \mid S(x)=g\}$, ${}_t p_x^{\overline{gg}} = \Pr\{S(x+u)=g\ \forall u \in [x,x+t] \mid S(x)=g\}$ and assume a closed system whereby $\sum_{h=1}^{6} {}_t p_x^{gh} = 1$ for all $x \geq 0$ and $t \geq 0$. The transition probabilities also obey the Chapman-Kolmogorov equations:

$${}_{t+u} p_x^{gh} = \sum_{l=1}^{6} {}_t p_x^{gl} \cdot {}_u p_{x+t}^{lh} \quad (3)$$

The existence of transition intensity functions is also assumed such that:

$$\mu_x^{gh} = \lim_{t \to 0+} \frac{{}_t p_x^{gh}}{t} \quad (4)$$

or, alternatively, that ${}_t p_x^{gh} = {}_t \mu_x^{gh} + o(t)$. Transition and occupancy probabilities are related to transition intensities via the relations:

$$\frac{d}{dt} {}_t p_x^{gh} = \sum_{l \neq h} \left({}_t p_x^{gl} \mu_{x+t}^{lh} - {}_t p_x^{gh} \mu_{x+t}^{hl} \right) \quad (5)$$

and

$${}_t p_x^{\overline{gg}} = \exp\left(-\int_0^t \sum_{l \neq g} \mu_{x+r}^{gl} dr \right) \quad (6)$$

where equations (5) are better known as the Kolmogorov forward equations. A more detailed discussion of Markov processes can be found in Cox and Miller (1965).

We further require the assumption that the transition intensities for each age in the 1998 ABS survey data are constant (ie. piecewise constant intensities). Consequently, if we define P(t) to be the matrix of transition probabilities over *t* years and Q to be the matrix of constant transition intensities per annum, then it can be shown directly from the Chapman-Kolmogorov equations (Jones (1992)) that:

$$P(t) = \exp(Qt) \tag{7}$$

Thus, calculating transition intensities requires finding the infinitesimal generator Q for the transition probability matrix P(t).

A number of numerical approaches may be used to determine Q such as uniformisation techniques (Stewart 1994) or the evaluation of Pade approximants (Higham 2001, Cheng et al 2001). We chose to use a Schur-Parlett method purely because of its straightforward implementation through software such as MATLAB, which we used.

The method, which is discussed in greater detail in Golub and Van Loan (1983), initially requires the computation of a Schur decomposition $\mathbf{P} = \mathbf{UTU}^*$, where U is a unitary matrix (ie. its entries are complex and its inverse is the conjugate-transpose), $\mathbf{U}^*$ is the conjugate transpose of U, and T is an upper triangular matrix. We can then determine functions of matrices (including natural logarithms) using the formula:

$$f(\mathbf{P}) = \mathbf{U}f(\mathbf{T})\mathbf{U}^* \tag{8}$$

Parlett (1974) proposes a recursive relationship for determining the matrix F, defined as *f*(T), which is derived from equating elements *(i,j)* where *i<j,*(ie. strictly upper triangular) in the commutivity relation FT=TF. The elements (*i,j*) in the commutivity result satisfy:

$$\sum_{k=i}^{j} f_{ik} t_{kj} = \sum_{k=i}^{j} t_{ik} f_{kj} \tag{9}$$

and as long as $t_{ii} \neq t_{jj}$ (ie. the eigenvalues are distinct), then:

$$f_{ij} = t_{ij} \frac{f_{jj} - f_{ii}}{t_{jj} - t_{ii}} + \frac{\sum_{k=i+1}^{j-1} \left[t_{ik} f_{kj} - f_{ik} t_{kj} \right]}{t_{jj} - t_{ii}} \tag{10}$$

Thus, starting with $f_{ii} = f(t_{ii})$, all other elements of F can be calculated one superdiagonal at a time. Tables 3 and 4 report the calculated annual transition intensities calculated from 1-step transition probabilities at 10-yearly age intervals for males and females respectively. As anticipated, there are a number of calculated transition intensities which are negative and thus have no physical interpretation. They remain useful, however, as starting values for our constraining algorithm in Section 6.2. We will refer to these 'unconstrained estimates' as $\tilde{\mu}_x^{ij}$ for the transition intensity at age x from state i to state j constituting matrix generator $\tilde{\mathbf{Q}}_x$.

We found the Schur-Parlett approach to give satisfactory results over the majority of the age range. We note, however, that the computational procedure was unstable at the extremely high ages. This is perhaps attributable to one or more of the following reasons:

The 1998 ABS survey data has ages beyond 85 grouped together in a single strata, thereby limiting our ability to understand the underlying process at higher ages.

The extremely high ages are the likely region where the assumption of constant intensities is most unrealistic.

Exposure at the higher ages is extremely low.

Table 3: Male unconstrained transition intensities calculated from 1-step transition probabilities in 10 yearly age intervals.

	Able	Mild	Moderate	Severe	Profound	Dead
Able						
20		0.005552	0.001896	0.000956	0.000830	0.001198
30		0.006628	0.002262	0.001141	0.000992	0.001308
40		0.009315	0.003172	0.001602	0.001395	0.001725
50		0.016034	0.005429	0.002746	0.002394	0.003511
60		0.032798	0.010957	0.005528	0.004817	0.010130
70		0.050588	0.016731	0.008445	0.007376	0.029426
80		0.172025	0.058298	0.031805	0.030252	0.078928
Mild						
20	0.163936		0.002300	0.001159	0.001007	0.001197
30	0.164190		0.002744	0.001384	0.001203	0.001306
40	0.164851		0.003850	0.001945	0.001693	0.001721
50	0.166644		0.006599	0.003337	0.002910	0.003498
60	0.171567		0.013363	0.006741	0.005874	0.010091
70	0.179329		0.020487	0.010338	0.009029	0.029352
80	0.225628		0.073590	0.040084	0.038093	0.078519
Moderate						
20	-0.014160	0.177510		0.001384	0.001202	0.001197
30	-0.014190	0.177686		0.001653	0.001436	0.001305
40	-0.014290	0.178158		0.002322	0.002022	0.001716
50	-0.014550	0.179494		0.003986	0.003476	0.003484
60	-0.015310	0.183309		0.008062	0.007025	0.010051
70	-0.016590	0.190219		0.012375	0.010808	0.029274
80	-0.024870	0.227448		0.048401	0.045992	0.078130
Severe						
20	0.001055	-0.010100	0.114858		0.001432	0.002916
30	0.001061	-0.010140	0.115073		0.001713	0.005416
40	0.001074	-0.010210	0.115556		0.002417	0.010587
50	0.001106	-0.010400	0.116579		0.004174	0.019538
60	0.001188	-0.010840	0.118786		0.008472	0.033564
70	0.001334	-0.011640	0.122760		0.013082	0.058420
80	0.002285	-0.015920	0.139842		0.056202	0.112765
Profound						
20	-0.000042	0.000364	-0.003120	0.054325		0.004500
30	-0.000043	0.000367	-0.003140	0.054540		0.009214
40	-0.000044	0.000373	-0.003180	0.054988		0.018817
50	-0.000046	0.000385	-0.003250	0.055764		0.034521
60	-0.000050	0.000411	-0.003380	0.056957		0.055611
70	-0.000058	0.000456	-0.003610	0.058828		0.085793
80	-0.000110	0.000680	-0.004480	0.064180		0.144751
Dead						
20	0	0	0	0	0	
30	0	0	0	0	0	

	Able	Mild	Moderate	Severe	Profound	Dead
40	0	0	0	0	0	
50	0	0	0	0	0	
60	0	0	0	0	0	
70	0	0	0	0	0	
80	0	0	0	0	0	

Table 4: Female unconstrained transition intensities calculated from 1-step transition probabilities in 10 yearly age intervals.

	Able	Mild	Moderate	Severe	Profound	Dead
Able						
20		0.005238	0.001357	0.000973	0.000954	0.000415
30		0.005998	0.001553	0.001115	0.001094	0.000486
40		0.008025	0.002075	0.001492	0.001467	0.000875
50		0.013420	0.003457	0.002492	0.002455	0.002177
60		0.027604	0.007069	0.005119	0.005065	0.005620
70		0.042664	0.011330	0.00860	0.008908	0.015295
80		0.123111	0.041157	0.040843	0.053228	0.047905
Mild						
20	0.163707		0.001728	0.001239	0.001215	0.000414
30	0.163882		0.001978	0.001419	0.001393	0.000484
40	0.164386		0.002643	0.001900	0.001868	0.000869
50	0.165780		0.004409	0.003178	0.003131	0.002161
60	0.169510		0.009048	0.006552	0.006483	0.005571
70	0.175197		0.014566	0.011055	0.011451	0.015189
80	0.217371		0.054786	0.054306	0.070644	0.047106
Moderate						
20	-0.014130	0.177359		0.001547	0.001517	0.000413
30	-0.014150	0.177491		0.001773	0.001740	0.000482
40	-0.014230	0.177885		0.002374	0.002334	0.000863
50	-0.014440	0.178994		0.003971	0.003913	0.002143
60	-0.015020	0.181963		0.008197	0.008110	0.005517
70	-0.015970	0.187150		0.013852	0.014349	0.015072
80	-0.024270	0.230334		0.069226	0.090010	0.046253
Severe						
20	0.001053	-0.010090	0.114811		0.001888	0.002131
30	0.001057	-0.010120	0.115016		0.002169	0.004590
40	0.001068	-0.010190	0.115491		0.002917	0.009727
50	0.001096	-0.010350	0.116458		0.004910	0.018176
60	0.001160	-0.010710	0.118329		0.010227	0.028925
70	0.001267	-0.011310	0.121394		0.018170	0.043837
80	0.002277	-0.016530	0.144654		0.116279	0.080699
Profound						
20	-0.000042	0.000364	-0.003120	0.054293		0.003715
30	-0.000043	0.000366	-0.003130	0.054505		0.008385
40	-0.000043	0.000372	-0.003170	0.054952		0.017952

	Able	Mild	Moderate	Severe	Profound	Dead
50	-0.000045	0.000383	-0.003240	0.055704		0.033144
60	-0.000049	0.000404	-0.003350	0.056724		0.050892
70	-0.000054	0.000436	-0.003520	0.058059		0.070886
80	-0.000110	0.000690	-0.004570	0.064107		0.112549
Dead						
20	0	0	0	0	0	
30	0	0	0	0	0	
40	0	0	0	0	0	
50	0	0	0	0	0	
60	0	0	0	0	0	
70	0	0	0	0	0	
80	0	0	0	0	0	

6.2 Constraining Transition Intensities to the Non-negative Region

Clearly we require transition intensities which are positive. We now discuss how we ensured this condition to produce 'constrained estimates', $\hat{\mu}_x^{ij}$, for the transition intensity at age x from state i to state j, constituting matrix generator $\hat{\mathbf{Q}}_x$.

Determining an appropriate method to deal with this requires care as adjusting negative 'transition intensities' to non-negative values will inevitably force other transition intensities, particularly those complementary to transition intensities that are negative, to compensate accordingly.

This problem has been encountered previously in the literature. For instance, Pritchard (2002) and Stallard and Yee (1999) both used US NLTCS data and estimated negative 'transition intensities' from transition probabilities. In this section, we outline four possible methods for constraining the transition intensities to be positive and discuss the approach ultimately pursued.

The most straightforward approach would simply be to set any negative 'transition intensities' to zero and compensate accordingly on the negative diagonal to retain a zero row sum. This was the approach adopted by Stallard and Yee (1999). Certainly this is the most computationally efficient approach. However, we decided against this method as we felt that our estimated negative transition intensities were not small enough to be forcefully disregarded entirely. Moreover, Stallard and Yee (1999) state that their small negative transition intensities should have been estimated as zero-values. There is no intuitive reason for this, however, in this study.

Pritchard (2002) similarly encounters the problem of negative transition intensities in his study of a disability model for LTC insurance using US NLTCS data. Pritchard (2002) calculates 2-year and 5-year transition probabilities using a maximum likelihood approach and transforms them into transition intensities using an inverted method from Section 6.4.2 of Kulkarni (1995). Pritchard (2002) then constrains the transition intensities to lie in the non-negative region by maximising the log-likelihood function and introducing a penalty function which ensures that all transition intensities remain non-negative during a computational maximisation procedure. We cannot implement such an approach in our study as the 1998 ABS data do not provide any information on the number or nature of transitions over a given period and thus do not allow a maximum likelihood approach for estimating transition probabilities or transition intensities.

We therefore restrict our attention to two possible approaches. The first originates from the mathematical finance literature relating to finding valid generators for credit rating matrices. Israel et al (2001) develop an algorithm for finding generators using Lagrange interpolation.

We implemented this approach by revisiting the relationship in (7) and estimating $\hat{\mathbf{Q}}$ using Israel et al's (2001) algorithm instead of our original Schur-Parlett method. Israel et al (2001) warn that the algorithm is inadequate when the eigenvalues $\theta_1, \theta_2, ..., \theta_n$ of P are 'close'. We found this inadequacy to cause the algorithm to fail for the vast bulk of our age range – particularly the young to mid age ranges. Tables 5 and 6 show the eigenvalues for the transition probability matrices estimated from 1998 ABS survey data at 10-yearly age intervals for both males and females respectively. We suspect that the 'close' eigenvalues causing the failure of the algorithm for most ages are the pairs (θ_1, θ_6) and (θ_2, θ_3) for both males and females. Even forcing the algorithm to consider up to 20 decimal places did not provide any improvement.

Table 5 **Male eigenvalues for transition probability matrices estimated from 1998 ABS survey data in 10-yearly age intervals.**

Age	Eigenvalue					
10	0.999821	0.838121	0.848222	0.896817	0.948796	1
20	0.998724	0.836277	0.846855	0.894568	0.945879	1
30	0.998483	0.834203	0.845930	0.891629	0.941491	1
40	0.997715	0.828986	0.843431	0.885246	0.932796	1
50	0.995003	0.815660	0.836371	0.872998	0.919348	1
60	0.986077	0.782183	0.816708	0.851791	0.901994	1
70	0.964480	0.737629	0.784764	0.821292	0.875627	1
80	0.534709	0.646476	0.721884	0.899341	0.811978	1

Table 6: **Female eigenvalues for transition probability matrices estimated from 1998 ABS survey data in 10-yearly age intervals.**

Age	Eigenvalue					
10	0.999838	0.839261	0.847249	0.895964	0.949215	1
20	0.999496	0.838461	0.846717	0.894471	0.947096	1
30	0.999287	0.837030	0.845997	0.891524	0.942841	1
40	0.998546	0.833052	0.843896	0.884956	0.934408	1
50	0.996327	0.822450	0.838032	0.872459	0.922028	1
60	0.990413	0.795729	0.821848	0.852858	0.908950	1
70	0.977043	0.760464	0.795421	0.828059	0.892909	1
80	0.567567	0.624489	0.706487	0.827405	0.915832	1

It would indeed be possible to modify the algorithm for the case of close or repeated eigenvalues (Singer and Spillerman (1976, Section 3.3b)). However, this search would be much more involved and more difficult to implement.

We choose instead to implement a simple constraining algorithm to constrain the transition intensities to lie in the non-negative region. That is, we estimate Q using

$$\hat{\mathbf{Q}} = \min_{Q_i}\left(\mathbf{P} - \exp(\mathrm{Q}_i)\right) \qquad (11)$$

where $\{Q_{1,} Q_2, \ldots\}$ are a set of matrices such that the elements *(i,,j)*, $i \neq j$, of $\hat{\mathbf{Q}}$ are non-negative. The matrix Q_1 is selected by the Solver routine in Excel, using the unconstrained transition intensities as starting values. Furthermore, $\exp(Q_i)$ is evaluated using a Taylor series expansion (Moler and Van Loan (1978) for series computations of matrix exponentials):

$$\exp(Q_i) = \sum_{z=0}^{\infty} \frac{1}{z!} Q_i^z = I_n + Q_i + \frac{1}{2!} Q_i^2 + \ldots + \frac{1}{z!} Q_i^z + \ldots \tag{12}$$

where I_n is the $n \times n$ identity matrix. An iterative built-in procedure in Excel generates Q_2, then Q_3 and so on until a suitable minimum is obtained.

Tables 7 and 8 show the annual constrained transition intensities calculated using the above algorithm at 10-yearly age intervals for both males and females respectively. Interestingly, the constraining procedure results in $\hat{\mathbf{Q}}$ having the non-negative off diagonal entries estimated as zero and recovery transitions only occurring progressively by one state – a likely feature of estimating transition intensities from transition probabilities estimated using Rickayzen and Walsh's (2002) framework.

Table 7 Male constrained transition intensities in 10 yearly age intervals.

	Able	Mild	Moderate	Severe	Profound	Dead
Able						
20		0.005535	0.001894	0.000956	0.000829	0.001198
30		0.006611	0.002262	0.001141	0.000991	0.001307
40		0.009299	0.003174	0.001603	0.001395	0.001725
50		0.016017	0.005431	0.002749	0.002399	0.003514
60		0.032777	0.010964	0.005540	0.004831	0.010139
70		0.050562	0.016749	0.008463	0.007401	0.029442
80		0.171942	0.058377	0.031924	0.030370	0.079030
Mild						
20	0.162740		0.002654	0.001267	0.001100	0.001293
30	0.162984		0.003072	0.001516	0.001317	0.001412
40	0.163619		0.004112	0.002139	0.001862	0.001864
50	0.165353		0.006806	0.003566	0.003138	0.003710
60	0.170124		0.013593	0.007001	0.006132	0.010323
70	0.177647		0.020748	0.010646	0.009331	0.029616
80	0.221915		0.074159	0.040752	0.038714	0.079062

	Able	Mild	Moderate	Severe	Profound	Dead
Moderate						
20	0	0.171054		0.000100	0.000100	0.000100
30	0	0.171502		0.000100	0.000100	0.000100
40	0	0.172716		0.000100	0.000100	0.000100
50	0	0.174649		0.001486	0.000970	0.001143
60	0	0.178058		0.005464	0.004424	0.007668
70	0	0.184316		0.009604	0.008040	0.026776
80	0	0.216654		0.044712	0.042458	0.075000
Severe						
20	0	0	0.111617		0.000100	0.000713
30	0	0	0.111888		0.000100	0.003300
40	0	0	0.112424		0.000368	0.008625
50	0	0	0.113317		0.002108	0.017608
60	0	0	0.115347		0.006328	0.031601
70	0	0	0.119016		0.010805	0.056369
80	0	0	0.134330		0.053389	0.110323
Profound						
20	0	0	0	0.053249		0.003661
30	0	0	0	0.053453		0.008374
40	0	0	0	0.053877		0.017969
50	0	0	0	0.054621		0.033667
60	0	0	0	0.055759		0.054741
70	0	0	0	0.057540		0.084883
80	0	0	0	0.062538		0.143705
Dead						
20	0	0	0	0	0	
30	0	0	0	0	0	
40	0	0	0	0	0	
50	0	0	0	0	0	
60	0	0	0	0	0	
70	0	0	0	0	0	
80	0	0	0	0	0	

Table 8: Female constrained transition intensities in 10 yearly age intervals.

	Able	Mild	Moderate	Severe	Profound	Dead
Able						
20		0.005219	0.001353	0.000973	0.000954	0.000416
30		0.005978	0.001551	0.001115	0.001094	0.000487
40		0.008004	0.002078	0.001491	0.001466	0.000875
50		0.013394	0.003471	0.002490	0.002454	0.002177
60		0.027578	0.007069	0.005126	0.005074	0.005631
70		0.042624	0.011333	0.008613	0.008928	0.015317
80		0.122889	0.041199	0.040990	0.053354	0.048004

	Able	Mild	Moderate	Severe	Profound	Dead
Mild						
20	0.162520		0.002092	0.001359	0.001336	0.000444
30	0.162691		0.002321	0.001562	0.001536	0.000522
40	0.163172		0.002919	0.002096	0.002062	0.000939
50	0.164520		0.004610	0.003409	0.003360	0.002345
60	0.168141		0.009264	0.006800	0.006728	0.005796
70	0.173658		0.014801	0.011343	0.011731	0.015443
80	0.214095		0.055272	0.055001	0.071246	0.047636
Moderate						
20	0	0.170766		0.000100	0.000100	0.000100
30	0	0.171140		0.000100	0.000100	0.000100
40	0	0.172247		0.000100	0.000100	0.000100
50	0	0.174135		0.001411	0.001344	0.000100
60	0	0.176850		0.005633	0.005546	0.003165
70	0	0.181535		0.011159	0.011670	0.012649
80	0	0.219895		0.065519	0.086646	0.043193
Severe						
20	0	0	0.111722		0.000100	0.000100
30	0	0	0.111999		0.000138	0.002566
40	0	0	0.112391		0.000892	0.007726
50	0	0	0.113222		0.002850	0.016234
60	0	0	0.114942		0.008102	0.026980
70	0	0	0.117768		0.015949	0.041836
80	0	0	0.138745		0.113418	0.078230
Profound						
20	0	0	0	0.053218		0.002867
30	0	0	0	0.053417		0.007542
40	0	0	0	0.053842		0.017105
50	0	0	0	0.054561		0.032294
60	0	0	0	0.055536		0.050029
70	0	0	0	0.056803		0.069999
80	0	0	0	0.062390		0.111520
Dead						
20	0	0	0	0	0	
30	0	0	0	0	0	
40	0	0	0	0	0	
50	0	0	0	0	0	
60	0	0	0	0	0	
70	0	0	0	0	0	
80	0	0	0	0	0	

We illustrate in Figures 4 and 5, the constrained transition intensities for both males and females from the able state estimated using the simple constraining algorithm described above.

Several important observations may be made here.

1. Clearly $\hat{\mathbf{Q}}_x$ no longer contains any negative 'transition intensities'.
2. The constraining procedure does not impact on the unconstrained negative intensities in isolation. All elements of the transition intensity matrix will be affected. However, a comparison of the unconstrained transition intensities against the resulting constrained transition intensities for both males and females reveals only marginal differences to other elements as a result of the constraining procedure.
3. We also note that several transition intensities have the tendency to change direction abruptly at the extremely high ages (eg. μ_x^{25} and μ_x^{34} for females).
4. Recovery intensities appear to be increasing as a function of age for both males and females. This initially seems counter intuitive. However, if we consider the conditional probability that a recovery transition occurs given a departure from the life's current state, it is easily verifiable that this quantity is indeed decreasing as a function of age - which is consistent with the underlying recovery process. A further reason lies with the Rickayzen and Walsh's (2002) feature of recovery transition probabilities which are constant for each age.

Finally, we note that although the constraining procedure produces a matrix $\hat{\mathbf{Q}}$ that has row-sums zero and non-negative off diagonal entries, it no longer satisfies P(1) = exp(Q) exactly. We are confident, however, that our constraining procedure which forcefully minimises the difference between P(1) and $exp(\tilde{\mathbf{Q}})$ produces a transition intensity matrix $\hat{\mathbf{Q}}$ closest to the true generator Q.

Overall, the method we use here to constrain the transition intensities is not critical as these intensities must ultimately be graduated in order to apply Thiele's differential equation approach, as will be discussed in the following section.

Figure 4: **Male constrained transition intensities.**

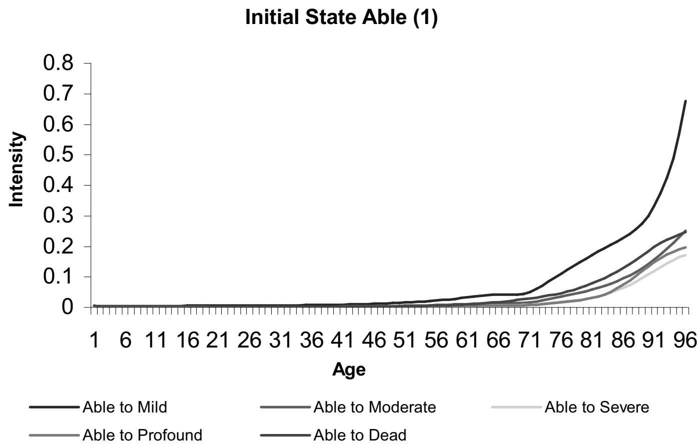
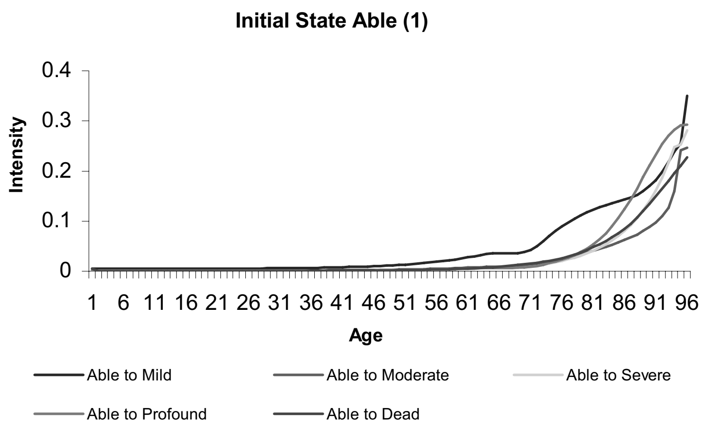


Figure 5: **Female constrained transition intensities.**



7.0 Graduating the Transition Intensities

In this section, we graduate the constrained transition intensities calculated in Section 6. We are restricted by our choice of graduation technique given our purposes here. Thus, graduation by mathematical formulae is pursued purely because of the need for functional forms for the constrained transition intensities for use in the Thiele's differential equations pricing and reserving framework. Graduation by mathematical formulae is discussed in detail in Benjamin and Pollard (1980), London (1985) and Forfar et al (1988). The graduation of transition intensities will be discussed here in three parts. We begin by graduating transition intensities to core activity restriction states, then graduate recovery transition intensities and finally graduate mortality transition intensities. Furthermore, smoothness and goodness of fit criteria are discussed here in relation to the absence of exposed to risk information.

7.1 Graduating Transition Intensities to Core Activity Restriction States

The transition intensities considered here are $\mu_x^{12}, \mu_x^{13}, \mu_x^{14}, \mu_x^{15}, \mu_x^{23}, \mu_x^{24}, \mu_x^{25}, \mu_x^{34}, \mu_x^{35}$ and μ_x^{45} for both males and females. Our choice of formulae was directly influenced by the functional forms used to estimate the original 1-step transition probabilities in discrete time from which these intensities were derived as discussed in Sections 5 and 6. We estimated transition probabilities to core activity restriction states according to a logistic type function motivated by Perks (1932) (Leung (2004)). We therefore chose to use a Perks formula specification, *a(x),* to graduate transition intensities to core activity restriction states. Moreover, we included an additional parameter, *H,* for the purposes of a more suitable fit.

$$a(x) = \frac{A + Bc^x}{1 + Dc^x + Kc^{-x}} + H \qquad (13)$$

Unsurprisingly, the Perks formula specification could not adequately fit the entire age range. We therefore blended an additional function for higher ages using a 5th order polynomial, *b(x)*. We found a 5th order polynomial sufficiently flexible to capture the dynamics in the raw information at the high ages while also retaining sufficient degrees of freedom in the specification. Note that the use of a simple polynomial function has been found in Continuing Mortality Investigation Report (CMIR) 7 (1984) to not necessarily result in inferior graduations as compared to more specialised formulae. We chose to blend the Perks formula and the 5th order polynomial at age 65. There were several cases, however, where the Perks specification was adequate to higher ages. In those cases, we blended the 5th order polynomial at age 90. We discuss our methods here assuming blending at age 65. The alternative is a trivial modification. As:

$$b(x) = \alpha_1(x-65)^5 + \alpha_2(x-65)^4 + \alpha_3(x-65)^3 + \alpha_4(x-65)^2 + \alpha_5(x-65) + \alpha_6 \quad (14)$$

the graduated transition intensities to core activity restriction states are specified as:

$$\overset{0}{\mu}{}_x^{ij} = \begin{cases} \dfrac{A+Bc^x}{1+Dc^x+Kc^{-x}}+H & x \le 65 \\ \alpha_1(x-65)^5 + \alpha_2(x-65)^4 + \alpha_3(x-65)^3 + \alpha_4(x-65)^2 + \alpha_5(x-65) + \alpha_6 & x \ge 65 \end{cases} \quad (15)$$

Smoothness was ensured to be adequate at the blending age by letting *a(65) = b(65)* and *a'(65) = b'(65)*. Note that one advantage of our polynomial specification of *b(x)* is that it allows us to satisfy these smoothness requirements easily. It is easily seen that *b(65)* and $b'(65)$ equals the parameter estimates of α_5 and α_6 respectively.

This leads to the next issue of parameter estimation. The parameters were calculated for the Perks function using unweighted non-linear least squares estimation to minimize the sum of squared errors, *SS,* between the observed and fitted intensities

$$SS = \sum_x \left[\hat{\mu}_x^{ij} - \overset{o}{\mu}{}_x^{ij} \right]^2 \quad (16)$$

Estimating parameters for the polynomial now becomes a straightforward least squares exercise. Given that α_5 and α_6 are already determined from *a'(65)* and *a(65)* respectively, taking partial derivatives of *SS* with respect to $\alpha_1, \alpha_2, \alpha_3$ and α_4 and equating to zero produces normal equations from which parameter estimates may be obtained. That is:

$$SS = \sum_x \left[\hat{\mu}_x^{ij} - \alpha_1 (x-65)^5 - \alpha_2 (x-65)^4 - \ldots - \alpha_6 \right]^2 \tag{17}$$

and

$$\frac{d}{d\alpha_y} SS = \sum_x \hat{\mu}_x^{ij} (x-65)^{6-y} - \alpha_1 \sum_x (x-65)^{11-y} - \alpha_2 \sum_x (x-65)^{10-y} - \ldots - \alpha_6 \sum_x (x-65)^{6-y} = 0 \tag{18}$$

for y=1,2,3,4.

We would have preferred to use a weighted least squares approach. However, this is not possible as there is no exposure information in our data.

We noted earlier that several transition intensities have the tendency to change direction abruptly at extremely high ages (eg. μ_x^{25} and μ_x^{34} for females). For these transition intensities, we found that the graduated curve behaved badly at the extremely high ages, sometimes producing negative values, and thus displaying instability in the graduation. This occurred very infrequently and only affected the last few ages in the age range. We therefore simply discarded these graduated rates. Again, this phenomenon and subsequent treatment is not uncommon in health related data (CMIR 7 (1984)). An alternative option was to graduate over a stable age range and extrapolate for other ages. We chose not to pursue this, but rather to adhere to our observed data. In any case, for our ultimate purpose of pricing and reserving calculations, we anticipate that the impact of a handful of transition intensities at the extremely high ages will be minimal. This will be confirmed once we test the sensitivity of the model (Section 9).

The parameter estimates for graduating transition intensities to core activity restriction states for both males and females by mathematical formula as specified in equation (14) are presented in Tables 9 and 10 respectively.

Table 9 Male parameter estimates for graduating transition intensities to core activity restriction states using a blended Perks and 5th order polynomial specification.

	Transition Intensity			
Parameters	$\mu_x^{o\,12}$	$\mu_x^{o\,13}$	$\mu_x^{o\,14}$	$\mu_x^{o\,5}$
A	0.001716	0.001192	-0.001740	0.001849
B	0.000112	0.000042	0.000032	0.000018
c	1.097952	1.093898	1.090271	1.097587
D	0.000127	-0.000048	-0.000190	0.000811
K	110.000000	110.000000	110.000000	110.000000
H	0.006186	0.002114	0.001342	0.000711
Blend Point	90.000000	90.000000	90.000000	65.000000
α_1	-0.000022	0.000141	-0.000002	-0.000000
α_2	0.000214	-0.001990	-0.000150	0.000003
α_3	-0.001560	0.009249	0.001942	-0.000041
α_4	0.015836	-0.013430	-0.007880	0.000191
α_5	0.019260	0.016841	0.019748	0.000402
α_1	0.321477	0.157799	0.124831	0.006495
Parameters	$\mu_x^{o\,23}$	$\mu_x^{o\,24}$	$\mu_x^{o\,25}$	$\mu_x^{o\,34}$
A	0.001762	0.002586	0.002470	-0.002080
B	0.000053	0.000027	0.000022	0.000045
c	1.093061	1.097779	1.097779	1.098795
D	-0.000100	0.001027	0.000907	0.002346
K	110.000000	110.000000	110.000000	110.000000
H	0.002894	0.001114	0.000945	0.000185
Blend Point	90.000000	65.000000	65.000000	65.000000
α_1	-0.000053	0.000000	-0.000000	0.000000
α_2	0.000467	-0.000008	0.000001	-0.000000
α_3	-0.000810	0.000175	0.000003	0.000012
α_4	0.002509	-0.001050	-0.000008	-0.000038
α_5	0.029229	0.000541	0.000478	0.000564
α_1	0.225823	0.009338	0.008153	0.008121

	Transition Intensity			
Parameters	$\mu_x^{o\,35}$	$\mu_x^{o\,45}$		
A	-0.002170	-0.001010		
B	0.000036	0.000048		
c	1.098811	1.099158		
D	0.002054	0.002476		
K	110.000000	110.000000		
H	0.000203	0.000119		
Blend Point	65.000000	65.000000		
α_1	-0.000000	-0.000000		
α_2	0.000002	0.000003		
α_3	-0.000018	-0.000035		
α_4	0.000039	0.000136		
α_5	0.000513	0.000559		
α_1	0.006751	0.009038		

Table 10 Female parameter estimates for graduating transition intensities to core activity restriction states using a blended Perks and 5th order polynomial specification.

	Transition Intensity			
Parameters	$\mu_x^{o\,12}$	$\mu_x^{o\,13}$	$\mu_x^{o\,14}$	$\mu_x^{o\,15}$
A	0.005823	0.001667	0.001234	0.001233
B	0.000125	0.000031	0.000022	0.000022
c	1.097723	1.097345	1.097263	1.097267
D	0.001315	0.001196	0.001006	0.000978
K	110.000000	110.000000	110.000000	110.000000
H	0.004864	0.001246	0.000901	0.000878
Blend Point	65.000000	65.000000	65.000000	65.000000
α_1	0.000000	0.000000	0.000000	-0.000000
α_2	-0.000003	-0.000001	-0.000005	0.000011
α_3	0.000031	0.000013	0.000095	-0.000180
α_4	0.000230	0.000085	-0.000520	0.001033
α_5	0.002226	0.000586	0.000452	0.000454
α_1	0.037427	0.009632	0.007070	0.007036

	Transition Intensity			
Parameters	$\overset{o}{\mu}{}_x^{23}$	$\overset{o}{\mu}{}_x^{24}$	$\overset{o}{\mu}{}_x^{25}$	$\overset{o}{\mu}{}_x^{34}$
A	0.000832	-0.006700	-0.019850	-0.010280
B	0.000041	0.000053	0.000080	0.000069
c	1.092977	1.090209	1.091327	1.089583
D	-0.000066	-0.000240	-0.000200	-0.000270
K	110.000000	110.000000	110.000000	110.000000
H	0.002054	0.002363	0.003899	0.001207
Blend Point	90.000000	90.000000	90.000000	90.000000
α_1	-0.000180	0.000342	-0.000065	-0.000350
α_2	0.003599	-0.008100	0.001607	0.008423
α_3	-0.021610	0.057726	-0.010930	-0.064790
α_4	0.031538	-0.090740	0.003431	0.142047
α_5	0.016561	0.050926	0.066463	0.075410
α_1	0.148599	0.253975	0.359350	0.338601
Parameters	$\overset{o}{\mu}{}_x^{35}$	$\overset{o}{\mu}{}_x^{45}$		
A	-0.002310	0.000922		
B	0.000044	0.000047		
c	1.098456	1.098481		
D	0.001818	0.001304		
K	110.000000	110.000000		
H	0.000196	0.000039		
Blend Point	65.000000	65.000000		
α_1	0.000000	-0.000000		
α_2	-0.000020	0.000024		
α_3	0.000372	-0.000500		
α_4	-0.001890	0.003365		
α_5	0.000681	0.000871		
α_1	0.008664	0.012035		

7.2 Graduating Recovery Transitions

The transition intensities considered here are those concerning recovery - $\mu_x^{21}, \mu_x^{32}, \mu_x^{43}$ and μ_x^{54}. Graduations using the Gompertz-Makeham and Logit Gompertz-Makeham formula of type (r,s) have been investigated previously using health and disability related data (for example CMIR 6 (1983) and CMIR 17 (1991)). Generally, the Logit Gompertz-Makeham formula is expressed as:

$$LGM_{\beta}^{r,s}(x) = \frac{GM_{\beta}^{r,s}(x)}{1 + GM_{\beta}^{r,s}(x)} \tag{19}$$

where:

$$GM_{\beta}^{r,s}(x) = \sum_{i=1}^{r} \beta_i x^{i-1} + \exp\left\{ \sum_{i=r+1}^{r+s} \beta_i x^{i-r-1} \right\} \tag{20}$$

is the Gompertz-Makeham formula of type *(r,s)* (Forfar et al 1985).

A Logit Gompertz-Makeham formula, $LGM(1,2)$, was found to fit sufficiently well here for recovery intensities, that is:

$$\overset{o}{\mu}{}_x^{ji} = \frac{\beta_1 + \exp(\beta_2 + \beta_3 x)}{1 + \beta_1 + \exp(\beta_2 + \beta_3 x)} \tag{21}$$

Female recovery transition intensities had the tendency to change direction abruptly at extremely high ages as discussed in Section 7.1. Note, however, that male recovery transition intensities did not have this problem. We chose to extrapolate over the higher ages for the female graduations. This was chosen purely to remain consistent with formulae used to graduate male recovery intensities. In any case, for our ultimate purpose of pricing and reserving calculations, we anticipate that the impact of this assumption will be minimal.

The parameters $\{\beta_1, \beta_2, \beta_3\}$ were estimated using unweighted least squares. The parameter estimates for graduating recovery transition intensities for both males and females by mathematical formula as specified in equation (21) are presented in Tables 11 and 12 respectively.

Table 11: Male parameter estimates for graduating recovery transition intensities using a $LGM(1,2)$ specification.

	Transition		Intensity	
Parameter	$\mathring{\mu}_x^{21}$	$\mathring{\mu}_x^{32}$	$\mathring{\mu}_x^{43}$	$\mathring{\mu}_x^{54}$
β_1	0.207171	0.215534	0.127122	0.056404
β_2	-30.050040	-26.272630	-17.137030	-11.934600
β_1	0.326204	0.282975	0.169227	0.094258

Table 12: Female parameter estimates for graduating recovery transition intensities using a $LGM(1,2)$ specification.

	Transition		Intensity	
Parameter	$\mathring{\mu}_x^{21}$	$\mathring{\mu}_x^{32}$	$\mathring{\mu}_x^{43}$	$\mathring{\mu}_x^{54}$
β_1	0.196390	0.223901	0.126574	0.055755
β_2	-19.623700	-67.149700	-25.843700	-13.576200
β_1	0.211458	0.744763	0.277626	0.118830

7.3 Graduating Mortality Transition Intensities

The final set of transition intensities to be considered are those concerning mortality - $\mu_x^{16}, \mu_x^{26}, \mu_x^{36}, \mu_x^{46}$ and μ_x^{56}. A $GM(2,2)$ model was found to fit sufficiently well here, that is:

$$\mathring{\mu}_x^{i6} = \gamma_1 + \gamma_2 x + \exp(\gamma_3 + \gamma_4 x) \tag{22}$$

Again, the parameters $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ were estimated using unweighted least squares. The parameter estimates for graduating mortality transition intensities for both males and females by mathematical formula as specified in equation (22) are presented in Tables 13 and 14 respectively.

Table 13: Male parameter estimates for graduating mortality transition intensities using a $GM(2,2)$ specification.

	Transition		Intensity		
Parameter	$\overset{o}{\mu}{}_x^{16}$	$\overset{o}{\mu}{}_x^{26}$	$\overset{o}{\mu}{}_x^{36}$	$\overset{o}{\mu}{}_x^{46}$	$\overset{o}{\mu}{}_x^{56}$
γ_1	0.006600	0.007678	0.007857	0.001235	-0.002113
γ_1	-0.000378	-0.000451	-0.000467	-0.000413	-0.000300
γ_1	-7.189564	-6.869178	-6.750137	-5.894103	-5.705457
γ_1	0.062122	0.058743	0.056746	0.050027	0.048946

Table 14: Female parameter estimates for graduating mortality transition intensities using a $GM(2,2)$ specification.

	Transition		Intensity		
Parameter	$\overset{o}{\mu}{}_x^{16}$	$\overset{o}{\mu}{}_x^{26}$	$\overset{o}{\mu}{}_x^{36}$	$\overset{o}{\mu}{}_x^{46}$	$\overset{o}{\mu}{}_x^{56}$
γ_1	0.002798	0.000394	0.004367	0.000805	-0.000210
γ_1	-0.000100	-0.000031	-0.000240	-0.000110	0.000378
γ_1	-10.388900	-10.127900	-7.660230	-7.069370	-8.447000
γ_1	0.094146	0.089905	0.061344	0.059315	0.075388

7.4 Smoothness and Goodness of Fit Criteria

One of the main advantages of graduating by mathematical formulae is that the resulting graduations are smooth. There is therefore no issue concerning smoothness here except in the case where two curves have been blended for graduating transitions to core activity restriction states. As already discussed, we have endeavoured to ensure a smooth transition across both curves by forcing endpoints of both curves to meet and first derivatives at end points to be equal.

Due to the non-existence of any exposed to risk data for our study, we were unable to use many of the conventional goodness of fit criteria such as the χ^2-test. We thus required some form of non-parametric goodness of fit measure. We chose to use the Theil

Inequality Coefficient (TIC) (Theil 1958) which is a scale invariant statistic typically used to assess econometric forecast samples. It is expressed as:

$$TIC = \frac{\sqrt{\frac{\sum_{x=0}^{n}\left(\overset{0}{\mu}_x - \hat{\mu}_x\right)^2}{n}}}{\sqrt{\frac{\sum_{x=1}^{n}\overset{0}{\mu}_X^{\,2}}{n}} + \sqrt{\frac{\sum_{x=1}^{n}\hat{\mu}_x^2}{n}}} \qquad (23)$$

and lies between 0 and 1 with 0 being a perfect fit. We accepted graduated curves with a coefficient of 10% or less. Table 15 reports the TIC for the graduated transition intensities for both males and females while Figure 6 illustrates the quality of the graduations for an indicative transition.

Overall, the reported TIC are generally low suggesting that the graduated curves provide a good fit to the observed transition intensities. Furthermore, it is interesting to note that the inequality coefficients for males appear to be better than the female counterparts despite there being no intuitive reason as to why this should occur. Three reported inequality coefficients for females ($\overset{o}{\mu}{}_x^{23}$, $\overset{o}{\mu}{}_x^{32}$ and $\overset{o}{\mu}{}_x^{36}$) are slightly greater than 10% suggesting that the formula specification for these transition intensities was sub-optimal. We chose not to change the formula specification for these three transition intensities and to retain consistency with the other intensities as the reported coefficients were only marginally greater than 10%.

Figure 6: Illustrative graduation and associated TIC Male:TIC = 0.01033

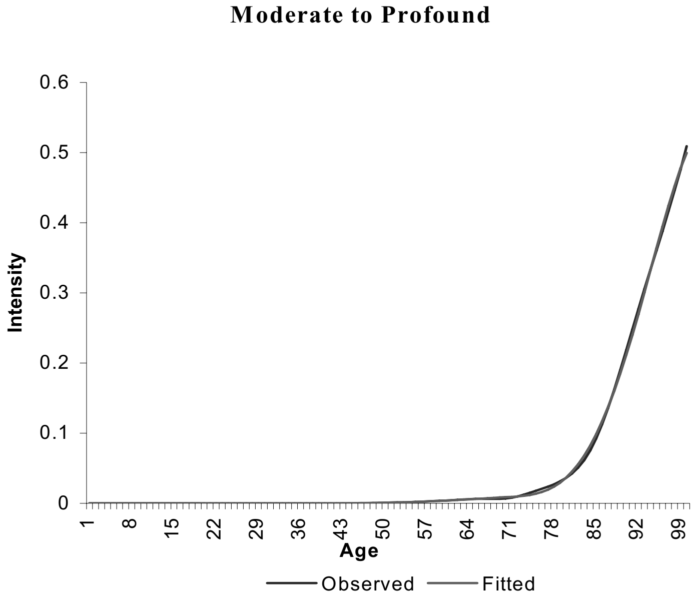


Table 15: Theil Inequality Coefficient (TIC) for graduated transition intensities.

	Male TIC	Female TIC
Transition Intensities to Core Activity Restriction States		
$\overset{o}{\mu}{}_x^{12}$	0.015170	0.034338
$\overset{o}{\mu}{}_x^{13}$	0.008610	0.031603
$\overset{o}{\mu}{}_x^{14}$	0.015140	0.023870
$\overset{o}{\mu}{}_x^{15}$	0.014060	0.025370
$\overset{o}{\mu}{}_x^{23}$	0.007220	0.106270
$\overset{o}{\mu}{}_x^{24}$	0.037410	0.020840
$\overset{o}{\mu}{}_x^{25}$	0.016320	0.053860
$\overset{o}{\mu}{}_x^{34}$	0.007250	0.091050
$\overset{o}{\mu}{}_x^{35}$	0.010330	0.010327
$\overset{o}{\mu}{}_x^{45}$	0.009110	0.031060
Recovery Transition Intensities		
$\overset{o}{\mu}{}_x^{21}$	0.062290	0.070650
$\overset{o}{\mu}{}_x^{32}$	0.040520	0.115090
$\overset{o}{\mu}{}_x^{43}$	0.013620	0.090540
$\overset{o}{\mu}{}_x^{54}$	0.007280	0.076640
Mortality Transition Intensities		
$\overset{o}{\mu}{}_x^{16}$	0.044690	0.017870
$\overset{o}{\mu}{}_x^{26}$	0.050630	0.043320
$\overset{o}{\mu}{}_x^{36}$	0.058010	0.111780
$\overset{o}{\mu}{}_x^{46}$	0.036310	0.065970
$\overset{o}{\mu}{}_x^{56}$	0.047450	0.022870

8.0 The Pricing and Reserving Framework

In this section, we turn to pricing and reserving. We discuss LTC benefit types and benefit triggers and the application of Thiele's differential equations as a framework for pricing and reserving LTC policies in Australia.

8.1 Long Term Care Benefit Types

Haberman and Pitacco (1999) describe four generic categories of LTC products. They are:

1. Fixed amount annuities (depending on disability level) sold to healthy people.
2. Fixed amount annuities (depending on disability level) sold to elderly people entering, or already residing in, residential care facilities.
3. Nursing and medical expense refunding.
4. Choice of fixed amount annuity or appropriate care service.

In this paper, only benefits of type 1 are considered. The pricing and reserving framework for other benefit types, from a technical perspective, become straightforward once an understanding of type 1 benefits is achieved. Haberman and Pitacco (1999) further outline several examples of products belonging to this category, including:

Stand-alone policy, providing a fixed amount annuity (possibly depending on frailty level) for persons requiring LTC.

LTC cover as a rider benefit, providing a fixed amount annuity (possibly depending on frailty level) for persons requiring LTC in addition to a whole life cover.

In this paper we consider both the stand-alone policy and the rider benefit policy.

8.2 LTC Benefit Triggers

A notable challenge associated with LTC products is the inherent difficulty in unambiguously defining a suitable benefit trigger for LTC claims.

Cowley (1992) outlines two distinct sets of eligibility criteria for claiming LTC benefits. Under the 'health insurance approach', benefit eligibility may be triggered by physician certified medical necessity or prescribed periods of pre-hospitalisation. Shortcomings to this approach, however, are that the criteria are subjective and exclude certain chronic cognitive impairments that do not generally call for periods of pre-hospitalisation before LTC is required.

The alternative 'disability insurance approach' is designed to be more objective – relying on a person's inability to perform certain ADLs as a benefit trigger. Such an approach also easily extends to include cognitive impairments and other chronic conditions. This approach is more useful for our purposes as the use of ADL failures is easily transposed onto a core activity restriction scale. For instance, a LTC benefit is typically paid upon failure of 3 or 4 ADLs. This may roughly be equivalent to a severe core activity restriction.

The 1998 ABS survey defines both severe and profound core activity restrictions to be levels of disability requiring assistance from another person. Assistance from another person is also a key aspect of the definition for LTC. We therefore require in our pricing and reserving framework that a life be either severely or profoundly restricted before being able to claim a LTC benefit.

8.3 Implementing Thiele's Differential Equation

The pricing and reserving methodology adopted here is essentially an application of Thiele's differential equations to derive formulae concerning the expected development of the mathematical reserve for a closed LTC insurance portfolio. The application of Thiele's differential equations to life contingencies (but not LTC) can be found in Hoem (1969), Hoem (1988), Linnemann (1993) and Norberg (1995).

We introduce some notation as follows. Let $V_i(r,u)$ denote the expected present value (EPV) of LTC benefits in the time interval *(r, u),* given that the policyholder is in state *i* at time *r* with a prevailing force of interest of δ over the period (r,u).

In general, for a multiple state model with *n* states, let $B_{jk}(t)$ denote the benefit payable at time *t* upon transition from state *j* to state *k,* and let $b_j(t)$ denote the rate of benefit payment at time *t* if the policyholder is in state *j*.

Then $V_i(r,u)$ may be expressed as:

$$V_i(r,u)=\int_r^u e^{-\delta(t-r)}\,{}_{t-r}p_{x+r}^{\overline{ii}}b_i(t)dt+\int_r^u e^{-\delta(t-r)}\,{}_{t-r}p_{x+r}^{\overline{ii}}\sum_{j\neq i}\mu_{x+t}^{ij}\left(B_{ij}(t)+V_j(t,u)\right)\mathrm{dt} \quad (24)$$

which leads to the generalisations of Thiele's differential equations:

$$\frac{d}{dr}V_i(r,u)=\delta V_i(r,u)-b_i(r)-\sum_{j\neq i}\mu_{x+r}^{ij}\left(B_{ij}(r)+V_j(r,u)-V_i(r,u)\right) \quad (25)$$

for $i = 1,2,\ldots,n$ (Hoem (1969)).

We now turn to pricing some illustrative LTC products.

8.4 Pricing Illustrative LTC Products

Consider first a whole of life stand-alone LTC policy where premiums are payable continuously at rate *P* per annum while the life is able (ie. no core activity restriction) and an annuity is payable to the policyholder at rate *A* per annum while enduring severe or profound core activity restrictions. That is, *A* per annum is paid to the policyholder when in need of LTC. Note that no death benefit is payable. For the purposes of premium calculation, we require the expected present value at time *t*=0 of a unit payment while the individual is in each of the able and LTC claiming states.

Therefore, consider first, the case where:

$$b_1(t) = 1,\ b_2(t) = b_3(t) = b_4(t) = b_5(t) = b_6(t) = 0 \text{, and } B_{ij}(t) = 0$$

for all *i* and *j* which allows us to calculate the present value of a unit payment, payable as long as the life is able – which ultimately translates to the calculation of premiums.

Thus we have the following equations:

$$\frac{d}{dr}V_1(r,u) = \delta V_1(r,u) - 1 - \Big[\mu_{x+r}^{12}\big(V_2(r,u) - V_1(r,u)\big) + \mu_{x+r}^{13}\big(V_3(r,u) - V_1(r,u)\big)$$

$$+\mu_{x+r}^{14}\big(V_4(r,u) - V_1(r,u)\big) + \mu_{x+r}^{15}\big(V_5(r,u) - V_1(r,u)\big)\Big] + \mu_{x+r}^{16}V_1(r,u)$$

$$\frac{d}{dr}V_2(r,u) = \delta V_2(r,u) - \Big[\mu_{x+r}^{21}\big(V_1(r,u) - V_2(r,u)\big) + \mu_{x+r}^{23}\big(V_3(r,u) - V_2(r,u)\big)$$

$$+\mu_{x+r}^{24}\big(V_4(r,u) - V_2(r,u)\big) + \mu_{x+r}^{25}\big(V_5(r,u) - V_2(r,u)\big)\Big] + \mu_{x+r}^{26}V_2(r,u)$$

$$\frac{d}{dr}V_3(r,u) = \delta V_3(r,u) - \Big[\mu_{x+r}^{32}\big(V_2(r,u) - V_3(r,u)\big) + \mu_{x+r}^{34}\big(V_4(r,u) - V_3(r,u)\big)$$

$$+\mu_{x+r}^{35}\big(V_5(r,u) - V_3(r,u)\big)\Big] + \mu_{x+r}^{36}V_3(r,u)$$

$$\frac{d}{dr}V_4(r,u) = \delta V_4(r,u) - \Big[\mu_{x+r}^{43}\big(V_3(r,u) - V_4(r,u)\big)$$

$$+\mu_{x+r}^{45}\big(V_5(r,u) - V_4(r,u)\big)\Big] + \mu_{x+r}^{46}V_4(r,u)$$

$$\frac{d}{dr}V_5(r,u) = \delta V_5(r,u) - \Big[\mu_{x+r}^{54}\big(V_4(r,u) - V_5(r,u)\big)\Big] + \mu_{x+r}^{56}V_5(r,u) \qquad (26)$$

Solving for $V_1(0,u)$ gives the expected present value of a unit payment to the individual while in the no core activity restriction state, say, EPV_1.

We also do the same for:

$$b_4(t) = 1,\ b_1(t) = b_2(t) = b_3(t) = b_5(t) = b_6(t) = 0 \text{, and } B_{ij}(t) = 0$$

for all *i* and *j* which allows us to calculate the EPV of a unit payment while the individual is in the severe core activity restriction state, say EPV_4.

$b_5(t)=1,\ b_1(t)=b_2(t)=b_3(t)=b_4(t)=b_6(t)=0$, and $B_{ij}(t)=0$

for all i and j which allows us to calculate the EPV of a unit payment while the individual is in the profound core activity restriction state, say EPV_5.

Using the principle of equivalence, the net annual premium, P, may be calculated as:

$$P\times EPV_1 = A\times\left(EPV_4+EPV_5\right) \tag{27}$$

Note that the system of Thiele's differential equations may not be solved analytically. We therefore solve numerically. Note also that u is required to be sufficiently large to mimic a whole of life assurance.

Furthermore, A and δ are flexible and may be modified easily. We provide a numerical example here (as an illustration and for comparative purposes) using the bases employed by Walsh and De Ravin (1995) and Walker (1990) who considered pricing LTC products in Australia using different modelling methodologies.

In these studies, a nominal rate of interest of 8% per annum is assumed where premiums are increasing at an assumed inflation rate of 4% per annum and benefits are similarly increased by 4% per annum whether the insured is claiming or not. Thus a 4% effective net interest rate per annum is appropriate for comparative purposes. A benefit level of $400 per week once in an LTC claiming state was also assumed for ease of comparison with Walsh and De Ravin (1995) and Walker (1990). Table 16 reports the net annual premium for a whole of life stand-alone LTC policy calculated at 5 yearly age intervals alongside results published by Walker (1990) and Walsh and De Ravin (1995).

Table 16: Male and female net annual premiums for a whole of life stand-alone LTC policy calculated using Thiele's differential equations compared to other studies.

Net Annual Premium ($ per annum): $400 per week LTC benefit

	Leung		Walker		Walsh & De Ravin	
Age	**Male**	**Female**	**Male**	**Female**	**Male**	**Female**
20	740	909	-	-	580	835
25	825	1043	-	-	706	971
30	937	1220	413	1030	850	1140
35	1084	1456	520	1314	978	1358
40	1283	1771	567	1702	1123	1648
45	1555	2200	834	2244	1349	2053
50	1931	2788	1090	3056	1706	2645
55	2457	3603	1516	4461	2306	3609
60	3212	4758	2399	7487	3557	5667

Several general comments can be made on the comparison of the results produced by this model and those of Walker (1990) and Walsh and De Ravin (1995).

Our results are consistent with previous studies in that male premium rates are uniformly less than female premium rates which is unsurprising given higher LTC utilisation rates by females.

Our results appear more closely in line with those of Walsh and De Ravin (1995). This is not surprising given the similarity in data source. Walsh and De Ravin (1995) base their premium rates on the 1993 ABS survey data – the survey immediately preceding the 1998 ABS survey used in this paper. Given that our rates are higher for both males and females as compared to Walsh and De Ravin (1995), it appears *prima facie* that our heavier premium rate is attributable to an increasing trend in disability. Note, however, that a change in survey design from 1993 to 1998 is well documented and no change in disability trend is apparent (Madden and Wen 2001). The difference in our premium rates is more likely attributable to Walsh and De Ravin (1995) only considering the profound core activity restriction category as an LTC claiming state whereas we consider both the profound and severe core activity restriction categories.

A further but less significant contributing factor may also be an improvement in mortality from 1993 to 1998.

Our results are higher than Walker (1990). The difference is undoubtedly related to the sources of data used. Walker (1990) restricts his attention to nursing home data. We would therefore expect that incidence rates used in Walker's (1990) premium calculations largely ignore LTC claims arising from non-institutional LTC and thus result in a lower premium.

Overall, the net annual premium rates for both males and females calculated using Thiele's differential equations within a multiple state model framework appear both reasonable and consistent with previous Australian studies.

This pricing framework may easily be extended to other LTC product types. For instance, consider a whole of life assurance policy with LTC rider benefit where premiums are payable continuously at rate P per annum while able (ie. no core activity restriction) and an annuity is payable to the policyholder at rate A per annum while enduring severe or profound core activity restrictions. In addition, a sum assured, S, is payable immediately on death from any live state.

That is, we need to consider the case where $b_i(t) = 0$ for $i = 1,2,...,5$ and $B_{j6}(t) = 1$ for j=1,2,…,5 which allows us to calculate the EPV of a unit payment when the individual transits to the dead state, say EPV_6.

Again, using the principle of equivalence, the calculation of the net annual premium for this rider benefit policy may be calculated as:

$$P \times EPV_1 = A \times \left(EPV_4 + EPV_5\right) + S \times EPV_6 \qquad (28)$$

Table 17 presents the net annual premium for a whole of life assurance policy with a LTC rider benefit, calculated at 5 yearly age intervals using the same basis as the stand-alone policy with a sum assured, S, of \$25 000.

Table 17: Male and female net annual premiums for a whole of life assurance policy with LTC rider benefit calculated using Thiele's differential equations.

Net Annual Premium ($ per annum): $400 per week LTC benefit, $25 000 death benefit

Age	Male	Female
20	917	1050
25	1041	1216
30	1211	1437
35	1442	1733
40	1758	2134
45	2193	2683
50	2800	3442
55	3658	4506
60	4897	6030
65	6767	8346

The premium rates for the whole of life assurance policy with LTC rider benefit are clearly heavier than the stand alone LTC policy reflecting the addition of the death benefit. Moreover, they are proportionally higher at the older ages as expected.

In contrast to net annual premiums, the single premium for a whole of life assurance policy with LTC rider benefit where premiums are payable continuously at rate P per annum while able (ie. no core activity restriction) and an annuity payable to the policyholder at rate A per annum while enduring severe or profound core activity restrictions with sum assured S, payable immediately on death from any live state, may be calculated directly by including all benefit payments and sums assured concurrently.

Table 18 presents the single premium for both a LTC stand alone policy and whole of life assurance policy with LTC rider benefit, calculated at 5 yearly age intervals using the same bases as per calculations for net annual premiums.

Table 18: Male and female single premiums for a whole of life assurance policy with LTC rider benefit and LTC stand-alone policy calculated using Thiele's differential equations.

Single Premium ($): $400 per week LTC benefit, whole of life policy includes $25 000 sum assured

	Stand Alone		Whole of life with rider	
Age	**Male**	**Female**	**Male**	**Female**
20	14457	18215	17570	20700
25	15492	20142	19136	23054
30	16646	22437	21103	25908
35	17930	25131	23258	29311
40	19346	28234	25803	33285
45	20874	31714	28616	37808
50	22464	35484	31603	42789
55	24039	39390	34635	48067
60	25560	43266	37609	53453
65	27222	47178	46661	58986

Given continuing advances in medical technology and declining mortality rates, the pricing of single premium business involves significant risk to the insurer. In practical terms, one would imagine that annual LTC contracts with rates adjustable by experience would find more favour among Australian insurers and reinsurers who may seek to hedge against improving morbidity experience.

8.5 Reserving for Illustrative LTC Products

Having solved for the net annual premium, *P,* we may calculate the development of the reserve for each state - $V_1(r,u), V_2(r,u), V_3(r,u), V_4(r,u)$ and $V_5(r,u)$. All we need specify are the boundary conditions given as:

$$V_1(u,u) = V_2(u,u) = V_3(u,u) = V_4(u,u) = V_5(u,u) = 0 \qquad (29)$$

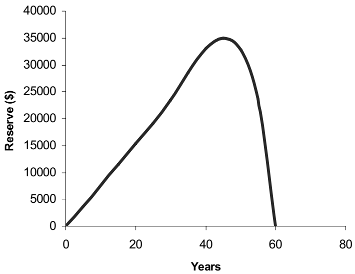
For illustrative purposes, we present results for the reserve profile for an insured male life aged 20 under a LTC stand-alone policy in Figure 6.

Overall, the results in Figure 7 show that the behaviour of $V_i(r,u)$ is largely as expected. Reserves for non-LTC claiming states ($V_1(r,u)$, $V_2(r,u)$ and $V_3(r,u)$) begin at zero (state 1) or a low level (states 2 and 3) and gradually build before falling and ultimately releasing the entire reserve at the end of the policy term. Reserves in LTC claiming states, however, begin at very high levels and gradually fall to zero at the end of the policy term.

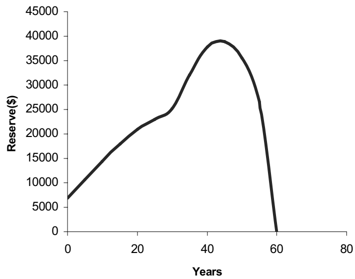
Insurers are likely to be most concerned with $V_1(r,u)$ as the vast majority of LTC policies would ordinarily be affected while the individual is in the non core activity restriction state. In each of the reserve profiles calculated here, $V_1(r,u)$ has a zero reserve at both contract issue and termination which is directly attributable to the equivalence principle. An interesting point to note is that the reserve levels for both the mild and moderate core activity restriction states ($V_2(r,u)$ and $V_3(r,u)$) begin at a positive non-zero level despite being a non benefit claiming state. This is a result of our hypothetical policy designs in this paper not requiring premiums to be paid while the insured is in either the mild or moderate core activity restriction states despite the probability of transiting to a LTC claiming state being greater than from the no core activity restriction state.

Figure 7: Reserve profile for a male insured life aged 40 under a LTC stand-alone policy.

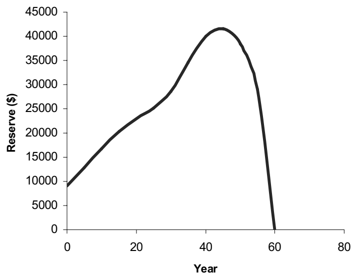
$V_1(r,u)$



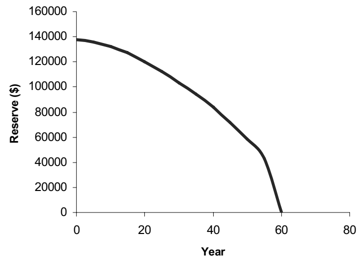
$V_2(r,u)$



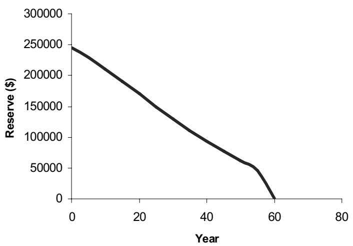
$V_3(r,u)$



$V_4(r,u)$



$V_5(r,u)$



9.0 Testing Model Sensitivities

Apart from the general diagnostic purposes of sensitivity testing of our model, there are two important additional motivating factors.

The constraining procedure in Section 6.2 to ensure the off-diagonal entries of the transition intensity matrices are non-negative implicitly causes deviation from the transition probabilities estimated from the data. We would therefore like to determine the impact of uniformly higher or lower transition intensities on our financial calculations.

The calculation of transition intensities from transition probabilities requires the assumption of piecewise constant intensities for each age. We suspect that this assumption is perhaps questionable at the extremely high ages as can be seen from several transition intensity functions abruptly changing direction at the last few ages (refer to Section 7). We anticipated that this would have a minimal impact on our financial calculations. We would like to determine if this is generally the case by uniformly adjusting transition intensities at the higher ages.

The approach in this section is to construct sixteen scenarios for both males and females with each scenario requiring modification to selected transition intensity functions. Premium calculations are then performed and compared to our 'best estimates' as determined in Section 8. For brevity, we restrict our attention here to the net annual premium calculated for the LTC stand-alone type policy.

The sixteen scenarios, A through P, are as follows:

A. Uniform 10% increase for all ages to transition intensities to LTC claiming states (ie. 10% increase to $\mu_x^{14}, \mu_x^{15}, \mu_x^{24}, \mu_x^{25}, \mu_x^{34}, \mu_x^{35}, \mu_x^{45}$).

B. Uniform 10% decrease for all ages to transition intensities to LTC claiming states (ie. 10% decrease to $\mu_x^{14}, \mu_x^{15}, \mu_x^{24}, \mu_x^{25}, \mu_x^{34}, \mu_x^{35}, \mu_x^{45}$).

C. Uniform 10% increase for all ages in mortality transition intensities from LTC claiming states (ie. 10% increase to μ_x^{46} and μ_x^{56}).

D. Uniform 10% decrease for all ages in mortality transition intensities from LTC claiming states (ie. 10% decrease to μ_x^{46} and μ_x^{56}).

E. Uniform 10% increase for all ages in recovery transition intensities from LTC claiming states (ie. 10% increase to μ_x^{54} and μ_x^{43}).

F. Uniform 10% decrease for all ages in recovery transition intensities from LTC claiming states (ie. 10% decrease to μ_x^{54} and μ_x^{43}).

G. Uniform 10% increase to transition intensities to LTC claiming states for lives aged 65 and over (ie. 10% increase to $\mu_x^{14}, \mu_x^{15}, \mu_x^{24}, \mu_x^{25}, \mu_x^{34}, \mu_x^{35}, \mu_x^{45}$ where $x \geq 65$).

H. Uniform 10% decrease to transition intensities to LTC claiming states for lives aged 65 and over (ie. 10% decrease to $\mu_x^{14}, \mu_x^{15}, \mu_x^{24}, \mu_x^{25}, \mu_x^{34}, \mu_x^{35}, \mu_x^{45}$ where $x \geq 65$).

I. Uniform 10% increase in mortality transition intensities from LTC claiming states for lives aged 65 and over (ie. 10% increase to μ_x^{46} and μ_x^{56} where $x \geq 65$).

J. Uniform 10% decrease in mortality transition intensities from LTC claiming states for lives aged 65 and over (ie. 10% decrease to μ_x^{46} and μ_x^{56} where $x \geq 65$).

K. Uniform 10% increase in recovery transition intensities from LTC claiming states for lives aged 65 and over (ie. 10% increase to μ_x^{54} and μ_{65+}^{43} where $x \geq 65$).

L. Uniform 10% decrease in recovery transition intensities from LTC claiming states for lives aged 65 and over (ie. 10% decrease to μ_{65+}^{54} and μ_x^{43} where $x \geq 65$).

M. Uniform 10% increase for all ages to transition intensities to LTC claiming states (ie. 10% increase to $\mu_x^{14}, \mu_x^{15}, \mu_x^{24}, \mu_x^{25}, \mu_x^{34}, \mu_x^{35}, \mu_x^{45}$), uniform 10% decrease for all ages in mortality transition intensities from LTC claiming states (ie. 10% decrease to μ_x^{46} and μ_x^{56}) and uniform 10% decrease for all ages in recovery transition intensities from LTC claiming states (ie. 10% decrease to μ_x^{54} and μ_x^{43}).

N. Uniform 10% increase to transition intensities to LTC claiming states for lives aged 65 and over (ie. 10% increase to $\mu_x^{14}, \mu_x^{15}, \mu_x^{24}, \mu_x^{25}, \mu_x^{34}, \mu_x^{35}, \mu_x^{45}$ where $x \geq 65$), uniform 10% decrease in mortality transition intensities from LTC claiming states for lives aged 65 and over (ie. 10% decrease to μ_x^{46} and μ_x^{56} where $x \geq 65$) and uniform 10% decrease in recovery transition intensities from LTC claiming states for lives aged 65 and over (ie. 10% decrease to μ_x^{54} and μ_x^{43} where $x \geq 65$).

O. Uniform 10% decrease for all ages to transition intensities to LTC claiming states (ie. 10% decrease to $\mu_x^{14}, \mu_x^{15}, \mu_x^{24}, \mu_x^{25}, \mu_x^{34}, \mu_x^{35}, \mu_x^{45}$), uniform 10% increase for all ages in mortality transition intensities from LTC claiming states (ie. 10% increase to μ_x^{46} and μ_x^{56}) and uniform 10% increase for all ages in recovery transition intensities from LTC claiming states (ie. 10% increase to μ_x^{54} and μ_x^{43}).

P. Uniform 10% decrease to transition intensities to LTC claiming states for lives aged 65 and over (ie. 10% decrease to $\mu_x^{14}, \mu_x^{15}, \mu_x^{24}, \mu_x^{25}, \mu_x^{34}, \mu_x^{35}, \mu_x^{45}$ where $x \geq 65$), uniform 10% increase in mortality transition intensities from LTC claiming states for lives aged 65 and over (ie. 10% increase to μ_{65+}^{46} and μ_x^{56} where $x \geq 65$) and uniform 10% increase in recovery transition intensities from LTC claiming states for lives aged 65 and over (ie. 10% increase to μ_x^{54} and μ_x^{43} where $x \geq 65$).

All scenarios involve modifications to transition intensity functions concerning LTC claiming states (severe and profound). Scenarios A to L include modifications to either transition intensities to LTC claiming states, mortality transition intensities from LTC claiming states or recovery transition intensities from LTC claiming states. Scenarios M to P involve a combination of modifications. From an insurer's perspective, scenarios M and N may be seen as the 'worst case' scenario and scenarios O and P as the 'best case' scenario.

Figures 8 and 9 illustrate the results of premium calculations for the LTC stand alone policy using the same bases as in Section 8 under each of the above scenarios for males and females respectively.

Figure 8: Male scenarios A to P

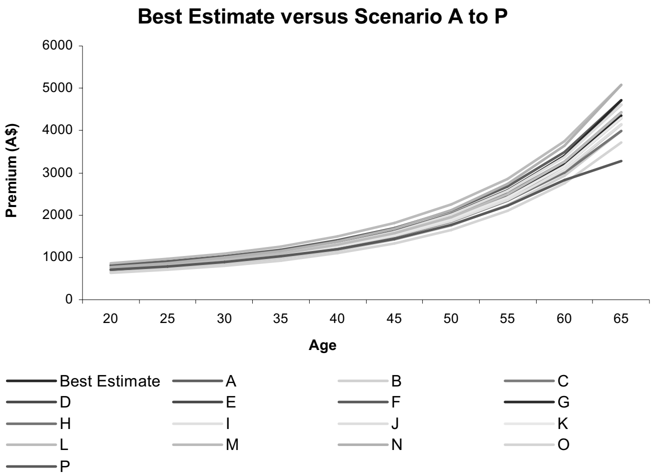
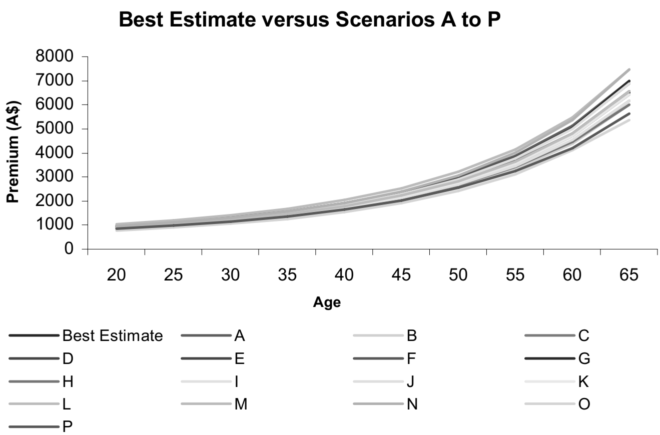


Figure 9: Female scenarios A to P



The results in Figures 8 and 9 show a reasonably narrow range of premium levels at each age for all 16 scenarios. This suggests that the impact of slightly different transition intensities to the intensities calculated as a result of the constraining procedure in Section 6.2 on premium and reserve calculations is relatively minimal. A further interesting observation is the results for scenarios G through L concerning modifications to transition intensities at the higher ages. Again, the range of premium levels at each age for these scenarios is reasonably narrow. The questionable assumption, therefore, of piecewise constant intensities at the higher ages appears to have all but a minimal impact on our financial calculations.

10. Conclusions and Further Research

In this paper, we develop a model for pricing and reserving LTC insurance using the currently available data in Australia – the 1998 ABS Survey of Disability, Ageing and Carers. We do this via the application of Thiele's differential equations for a multiple state model. This model, despite its complexity, offers a significant degree of modelling flexibility and robustness which makes it preferable to traditional annuity inception approaches. This study, to our knowledge, represents the first stochastic model developed for the purposes of pricing and reserving LTC in Australia.

There are, however, a number of limitations here. In particular, we acknowledge the inconsistency of a continuous time Markov chain with the discrete model framework of Rickayzen and Walsh (2002). We have managed to construct a model, however, that largely circumvents the difficulties introduced here along with a set of sensitivity scenarios in response to this constraint. Our focus to a great extent in this paper has been the development of a model which, when adequate data becomes available, will produce increasingly accurate results.

Moreover, once appropriate LTC specific data become available for Australia, the following extensions to this study could be undertaken:

1. Allow for all possible modes of recovery in the multiple state model;
2. Allow for lapses in the multiple state model.
3. Allow for duration in the multiple state model by implementing a semi-Markov assumption as opposed to a Markov assumption. That is, allow transition intensities to depend on both age and the duration of stay in the current state.

For comparative purposes, the technical actuarial bases used to price and reserve several illustrative LTC products come from a survey of earlier relevant Australian literature. The model bases here, however, may be easily modified at the insurer's discretion.

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Discussion
A Multiple State Model for Pricing and Reserving Private Long Term Care Insurance Contracts in Australia

Melbourne

Darryl MacKay BSc, FIAA

Eddie, look well done. I thought you spoke very well today and even more so given the noise outside, so congratulations on that. I'm not sort of qualified to comment on your approach and things like that. But I do have a couple of questions. The initial data that you got from the ABS, my first question is, were you surprised by any of that data? For example one of the things you, one of the age groups you had there, I think it was 5 to 14, the able category was only like 911 out of 1000. I was surprised how low that was. I would just be interested if you could comment on that. And secondly I'd be interested in your comment, one comment you've got on why long-term care hasn't actually taken off in Australia?

Edward Leung BCom (Hons), LLB (Hons), PhD, AIAA

The two aspects of the data that were surprising were the extreme ages - that is, the very early ages 0-4, perhaps 5-15 and also ages 85 and above. Clearly there is some scope for systemic error within the data. Firstly, it is critical to note that it is self-reporting data. We are therefore relying on respondents to determine their own functional capacities with respect to those three core activities restrictions that I mentioned - ie self-care, mobility and communication. Secondly there is an element of discretion as to how those responses are categorised into the four core activity restriction categories. As a result, non-intuitive data points, particularly at the extreme ages, may be a result of issues in self reporting or discrepancies in categorisation of responses to the various levels of core activity restriction.

With respect to why there has been limited long term care business written in Australia, it is worth noting that long-term care policies were actually piloted in the late 1980s in Australia but the extent of their market penetration was about 200 policies. My view is that there are several reasons for this. Perhaps most obvious is that unlike the US and UK, public subsidy for long-term care in Australia is extensive which perpetuates the view that 'if I need long term care, the government will take care of me'.

Andrew Gower BEng (Hons)

Firstly thanks for doing this. This will obviously be of use as the population does age over the next probably 20 or 30 years and the Government probably will withdraw some of those subsidies or minimize them. I guess two questions, firstly around the data. Did you actually ask the ABS if they could, if you could actually see the raw data or just get the summary that they obviously can provide publicly? I image the raw data would be at least have age based information rather than age bands.

Secondly around the model you used. You used the full; you put four full states of care. In reality the pricing that you did, you just said the benefit is paid if you're in the profound or severe. Why didn't you just do a three state model, which we use in Life Insurance at least for income protection and similar things? Because that would at least make it an easier proposition to tackle in the real world.

Edward Leung - Response

To answer the first question with respect to the data; the data that I used was the published data. I did not have access to the full set of results and I did not have access to any of the raw data or surveys. The point that you make with the age bands is significant.

As I mentioned, the age band 85 and above is material. My model relies on understanding the dynamics on an age-by-age basis. However, from ages 85 and above we don't really have a good idea from the data as to what the underlying dynamic actually looks like despite it being the most important age band for pricing a long term care risk.

Since the ABS have only published the very high ages in one band, all that we can really do is extrapolate what we think is going to happen at those very high ages – which is what I have done. In terms of your second question as to why not just use three states?

The reason for that is that I felt that it was necessary to consider transitions to and from the mild and moderate state as opposed to just the able state. To give you an example, if someone was in say the severe state for one period, improves one state to the moderate state but then in the following period moves back, that is a very different proposition to say recovering fully to able and then going back to the severe or profound state. So it really comes down to a question of how realistic we could make the underlying process look like.

Tim Gorst MEc, FIAA

Thanks for your paper Eddie, just a couple of points around the practical application of this theory. Long term care is obviously a massive issue for the Government and for the public going forward and it would be great to see how some of your theory might translate into estimates of what will be the public burden of this issue going forward. That could in turn raise some profile on what will likely become a big reserving issue for the Government. Another couple of potential applications are in the space of financial planning, where financial planners are asked to advise on what is a suitable level of income for someone to retire on. An extension of your work might highlight that long term care is a material chunk of someone's retirement benefit that they need to put aside for.

Finally in terms of long-term care product sales, why aren't they taking off? On reflection, there is a product out there at the moment to help with the situation of long term care and that's the reverse mortgage products. I think they're going to potentially find a big market for people who suddenly need to draw down on their home equity to fund for some sort of long term care. Perhaps there could be an application of the theory that you've brought forward here today to the future product design of reverse mortgage products.

Edward Leung - Response

I would like to touch upon your final point about the reverse mortgages and home equity. It's interesting that you make the point of the potential link between long term care and home equity because that is in fact something that is actively being pursued in jurisdictions such as Canada. So there is some precedence and historical experience for us out there.

Martin Stevenson BSc, FIA, FIAA

Thank you to everyone who has come along and who has contributed to the discussion. May I also convey a special thank you to Eddie Leung for a high quality paper that has added considerably to the Institute's intellectual property. Please record your appreciation to Eddie in the usual way.

Sydney

Stuart Rodger BA, FIA, FIAA

First of all Edward, congratulations on (a) doing the work and (b) taking the trouble to present it to the Profession. Once it's at a session like this, it's in our literature for all time. So that's especially why I thank you.

My question is that as I understand it, you've looked at a specific set of data based on recent experience and come up with rates that reflect that data; and one of the potential extensions of the work is about when better data becomes available. One of the types of better data that would become available is material that reflects the change in population experience over time. I expect that for long term care, the experience in the previous decade, is not necessarily the same as that for the next decade. That means when you design a product around this, you need to be able to deal with that change in the underlying claims experience.

My question is: In the work that you did for this, did a model for how to structure the potential changes in a product come to mind as well? Let me explain. For obvious reasons you've illustrated the concepts with an inflating premium, whole of life contract, terms set at outset, for which you can show the required premium. However, as the experience underlying changes, how would you see the product being able to be changed once sold?

Edward Leung BCom (Hons), LLB (Hons), PhD, AIAA

Unfortunately, I am unaware of any papers that have dealt with how the design of a long-term care product would change given historical experience. The primary reason for this, at least in the Australian context in my view, is that there is simply limited Australian literature on long term care aside from a handful including Walker, Walsh and De Ravin and my own. In terms of any relevant literature that has emerged overseas - I am not aware of any. I suspect that the best source for this information would emerge either from the US or the UK given the maturity of their respective long term care systems.

Stuart Rodger

Perhaps a further comment: I worked for a while in the UK for a company that offered long term care insurance and one of our concerns about the underlying data was whether in this area the past is any guide to the future.

In designing our products we reserved to ourselves the right to change the risk premium that we were charging for claims experience, and that led us to design products where there was an unbundling between the amount invested in the policy and the risk cost of the cover each year. That was enough to change us from not wanting to sell the product to at least being able to consider selling the product. To that extent this design was a good thing, but of course it left the consumer carrying more risk. That was why I was interested in whether this exposition led you to a different mechanism for controlling changes in underlying experience.

Martin Stevenson BSc, FIA, FIAA

My understanding is that policies of this nature either aren't written in Australia or are extremely rare. Is there any reason for the lack of them in Australia? Have people tried it? Is this something about our social security system or cultural environment, which means they haven't been popular to date?

Edward Leung - Response

Private long term care business has been unsuccessfully piloted in Australia. My understanding is that two companies launched long term care policies in the late 1980s. The extent of their market penetration, however, was only 200 policies. The obvious reason why such policies haven't taken off in Australia is that the current level of Commonwealth Government subsidy for long-term care is high. Looking forward, it is clearly going to be politically unsettling and therefore difficult to withdraw these subsidies.

Stuart Rodger

I can inject a bit of the UK experience into that as well. Our company had a large share of the market at that time. And we had a relatively small number of independent financial advisors advising consumers on this product, out of the many hundreds of IFAs that exist in the UK. Those few advisors were very switched on to the product, and I've observed a market research video of their customers.

The consumers who bought from the IFAs were very switched on to the risk, but nevertheless the conclusion we drew was that the risk itself and the need itself is a very difficult one to sell to the consumer, making it difficult for spending on these premiums to be more important than spending on other things.

Eddie Leung - Response

I would agree with your second comment. My view is that consumers would consider long term care to be a risk very far out in the future or simply a risk they don't want to contemplate.

Stuart Rodger

For the record, while we've talked about how difficult a sale it is in the UK and Australia, there are some places in the world where there's a lot of people taking this product up, so it is not all bad news for long term care.

Eddie Leung - response

Overall, I think that is a fair comment. Are you referring to the United States where people are taking the product up?

Stuart Rodger

Yes, also there's quite good coverage of the product in Singapore.

Martin Stevenson

I agree very much with Stuart Rodger's comments. This is a very worthwhile paper and we appreciate the effort that has gone into producing it. Thank you all for attending and once again, thank you Eddie for your paper.

Notes

2006 VOLUME 12
issue 2

Premiums and Reserves for Life Insurance Products

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1 Introduction

In recent years multiple state models have been used to tackle a variety of actuarial problems, for example in the study of retirement communities (Jones (1996)), in the study of the effects of genetic testing on insurers (Macdonald (1997)), and in costing long term care (Leung (2006)).

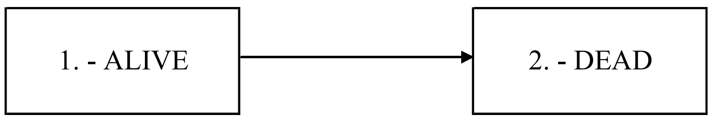
The current Australian Part I education course introduces students to multiple state models, but provides no guidance on how to perform calculations in anything other than a very simple setting. The purpose of this paper is to show how calculations can be performed for multiple state models using mathematical software. We illustrate not only how to calculate probabilities under a multiple state model, but also how to calculate premiums and reserves.

We start in Section 2 by giving some illustrations of multiple state models. In Section 3 we then give formulae from which probabilities under a multiple state model can be calculated, and illustrate their application. We then move to pricing and reserving for life insurance products in Section 4. Throughout this paper we perform calculations using the software Mathematica.

2 Examples of multiple state models

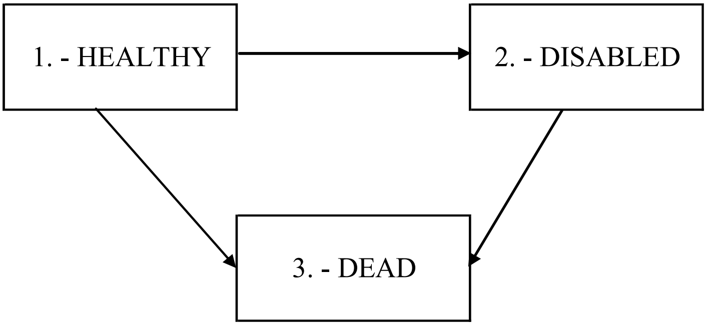
The simplest possible example of a multiple state model is the two state "alive-dead" model, illustrated in Figure 2.1. Under this model, an individual starts in the live state and transfers at some random future time to the dead state.

Figure 2.1: The "alive-dead" model



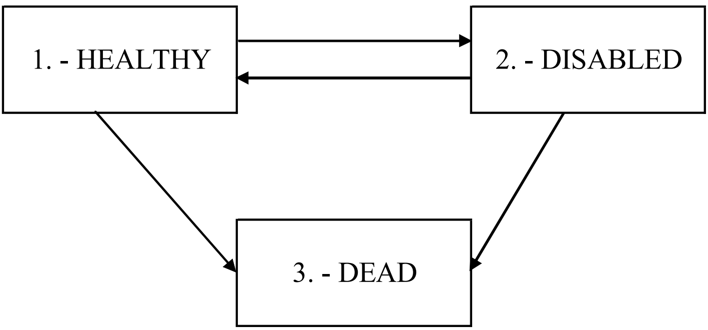
A slightly more complicated model is what we might call a permanent disability model, illustrated in Figure 2.2. Under this model, an individual starts in the healthy state and can move to the disabled state or to the dead state. If a move takes place to the disabled state, the only possible future move is to the dead state.

Figure 2.2: The permanent disability model



A feature of these first two examples is that once an individual leaves a state, a return to that state is not possible. In our third example, which we refer to as the disability income insurance (DII) model, if a move takes place from the healthy state to the disabled state, a return to the healthy state is possible, as illustrated in Figure 2.3. Indeed, multiple moves may take place between the healthy and disabled states.

Figure 2.3: The disability income insurance model



3 Probabilities for a multiple state model

3.1 Mathematical description

We start by considering a very general model which has n states, labelled $1,2,...,n$. For all g and h define

$$ {}_t p_x^{gh} = \Pr(\text{life is in state } h \text{ at age } x+t | \text{life is in state } g \text{ at age } x) $$

and define

$$ {}_t p_x^{\overline{hh}} = \Pr(\text{life is in state } h \text{ continuously from age } x \text{ to age } x+t). $$

Central to multiple state models are transition intensities between states. The transition intensity at age x from state g to state h (where $g \neq h$) is defined as

$$ \mu_x^{gh} = \lim_{t \to 0^+} {}_t p_x^{gh} / t. $$

Alternatively, we may write

$$ {}_t p_x^{gh} = t \mu_x^{gh} + o(t) $$

where a function $f(t)$ is said to be $o(t)$ if

$$ \lim_{t \to 0} \frac{f(t)}{t} = 0. $$

In the "alive-dead" model, the familiar actuarial terms ${}_t p_x$, ${}_t q_x$ and μ_x become ${}_t p_x^{11}$, ${}_t p_x^{12}$ and μ_x^{12}. We also note that under this model

$$ {}_t p_x^{11} = {}_t p_x^{\overline{11}} $$

since an individual who is in the alive state at age x is in that state at age $x+t$ only if no move takes place out of that state.

3.2 The Kolmogorov equations

The Kolmogorov forward equations are what we use to calculate probabilities under a multiple state model. They are well documented and can be found in most standard texts on stochastic processes, for example Ross (1996). In our general model these equations are

$$\frac{\partial}{\partial t}\, {}_t p_x^{gh} = \sum_{j \neq h} \left({}_t p_x^{gj} \mu_{x+t}^{jh} - {}_t p_x^{gh} \mu_{x+t}^{hj} \right)$$

and

$$\frac{\partial}{\partial t}\, {}_t p_x^{\overline{hh}} = -\, {}_t p_x^{\overline{hh}} \sum_{j \neq h} \mu_{x+t}^{hj},$$

which leads to

$${}_t p_x^{\overline{hh}} = \exp\left(-\int_0^t \sum_{j \neq h} \mu_{x+r}^{hj}\, dr \right). \qquad (3.1)$$

In actuarial work (and notation) the best known special case of these equations is probably

$$\frac{d}{dt}\, {}_t p_x = -\, {}_t p_x \mu_{x+t}$$

which applies to the "alive-dead" model. For this model, equation (3.1) is simply

$${}_t p_x = \exp\left(-\int_0^t \mu_{x+r}\, dr \right).$$

The transition intensities are fundamental to modelling. Given the transition intensities, we can use the Kolmogorov equations to solve numerically for probabilities under any multiple state model.

Example 3.1: *For the DII model, for a life in State 1 at age* x*, the above general results become*

$$ {}_t p_x^{\overline{11}} = \exp\left(-\int_0^t \left(\mu_{x+r}^{12} + \mu_{x+r}^{13}\right) dr\right), $$

$$ \frac{d}{dt}\,{}_t p_x^{11} = -\left(\mu_{x+t}^{12} + \mu_{x+t}^{13}\right) {}_t p_x^{11} + \mu_{x+t}^{21}\,{}_t p_x^{12} $$

and

$$ \frac{d}{dt}\,{}_t p_x^{12} = -\left(\mu_{x+t}^{21} + \mu_{x+t}^{23}\right) {}_t p_x^{12} + \mu_{x+t}^{12}\,{}_t p_x^{11}. $$

Similar equations apply if the life is in State 2 at age x.

3.3 Numerical implementation

Example 3.2: *For the DII model, let us suppose that*

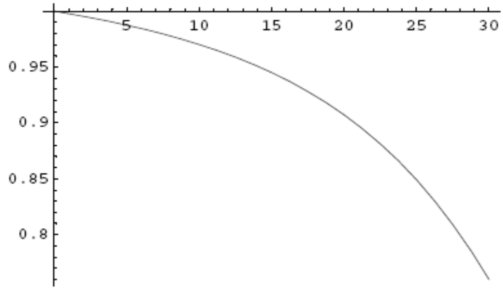
$$ \begin{aligned} \mu_x^{13} &= \mu_x^{23} = 0.0005 + 0.000075858 \times 10^{0.038x}, \\ \mu_x^{12} &= 0.0004 + 0.0000034674 \times 10^{0.06x}, \\ \mu_x^{21} &= 0.005. \end{aligned} $$

These values are taken from Norberg (1995) who cites the first three as values used by Danish insurance companies for male lives.

The output below shows how we can solve for ${}_t p_{30}^{1j}$*, for* $j = 1,2,3$ *using Mathematica, and hence calculate annuity values.*

```
In[1]:= μ[x_] :=
         0.0005 + 0.000075858 * (10 ^ (0.038 * x))

        (* μ[x] is μx^13 *)

In[2]:= ν[x_] :=
         0.0005 + 0.000075858 * (10 ^ (0.038 * x))

        (* ν[x] is μx^23 *)

In[3]:= σ[x_] :=
         0.0004 + 0.0000034674 * (10 ^ (0.06 * x))

        (* σ[x] is μx^12 *)

In[4]:= ρ = 0.005
        (* ρ is μx^21,
         independent of x in this example *)

Out[4]= 0.005

In[5]:= NDSolve[
         {P'[t] == -(σ[30 + t] + μ[30 + t]) * P[t] +
            ρ * Q[t], Q'[t] ==
           -(ρ + ν[30 + t]) * Q[t] + σ[30 + t] * P[t],
          P[0] == 1, Q[0] == 0}, {P, Q}, {t, 0, 30}]

Out[5]= {{P → InterpolatingFunction[{{0., 30.}},
            <>], Q → InterpolatingFunction[
            {{0., 30.}}, <>]}}

        (* P[t] represents tpx^11
          and Q[t] represents tpx^12 *)
        (* Output is stored in
          interpolating functions *)
```

```
In[6]:= Plot[P[t] /. %5, {t, 0, 30}]
```

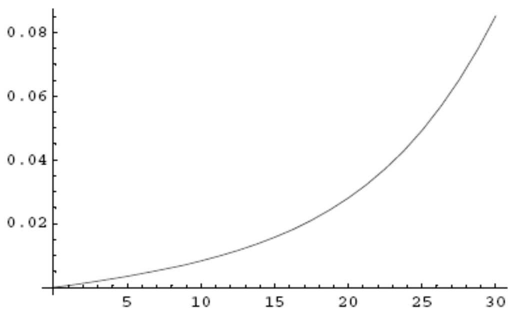



```
Out[6]= - Graphics -

(* the /.%5 refers to P[t] of Out[5] *)

In[7]:= TableForm[
         Table[{t, P[t] /. %5}, {t, 0, 30, 5}]
```

Out[7]//TableForm=

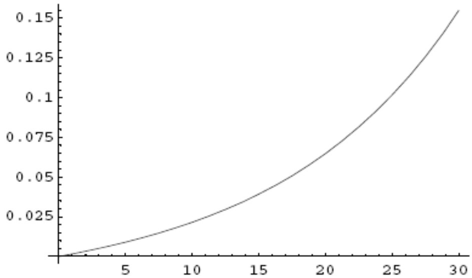
0	1.
5	0.987477
10	0.970182
15	0.945071
20	0.907247
25	0.849132
30	0.760049

```
In[8]:= Plot[Q[t] /. %5, {t, 0, 30}]
```




```
Out[8]= - Graphics -
```

```
In[9]:= Plot[1 - P[t] - Q[t] /. %5, {t, 0, 30}]
```




```
Out[9]= - Graphics -

In[10]:= δ = Log[1.045]

Out[10]= 0.0440169

In[11]:= Sum[e^(-δ t) P[t] /. %5, {t, 0, 29.75, 0.25}] / 4

Out[11]= {15.8626}

(* This is the EPV at force of interest
   4.5 % pa of an annuity of 1 payable
   quarterly in advance for 30 years
   as long as (30) is in State 1 *)
```

4 Insurance mathematics

4.1 Equations

We again start by considering a general model in which there are n states. Let $V_i(r,u)$ denote the expected present value (EPV) of benefit outgo at force of interest δ per annum in the time period (r,u) given that the policyholder is in state i at time r. (Benefits can be negative valued, i.e. income to the insurer.) Let x denote the policyholder's age at issue.

In what follows, we will think of $V_i(r,u)$ as a net premium reserve, assuming $V_1(0,\infty) = 0$.

Example 4.1: *In the two state "alive-dead" model, suppose that net premiums (negative benefits) are payable continuously at rate* P *per annum, and that a sum assured of* S *is payable immediately on death. Then*

$$V_1(0,\infty) = -P\int_0^\infty e^{-\delta t}\, {}_t p_x^{\overline{11}} dt + S\int_0^\infty e^{-\delta t}\, {}_t p_x^{\overline{11}}\, \mu_{x+t}^{12}\, dt.$$

Setting $V_1(0,\infty)=0$ *makes* P *the net premium. We can, of course, arrive at the above without thinking in terms of multiple state models. In standard actuarial notation, the above equation is*

$$\begin{aligned} V_1(0,\infty) &= -P\int_0^\infty e^{-\delta t}\, {}_t p_x dt + S\int_0^\infty e^{-\delta t}\, {}_t p_x\, \mu_{x+t}\, dt \\ &= -P\bar{a}_x + S\bar{A}_x. \end{aligned}$$

Example 4.2: *In the permanent disability model, suppose that net premiums are payable continuously at rate* P *per annum while a life is in the healthy state, an annuity is payable to the policyholder at rate* A *per annum while disabled, and a sum assured of* S *is payable immediately on death (from either live state). Then*

$$V_2(r,\infty) = A\int_r^\infty e^{-\delta(t-r)}\, {}_{t-r} p_{x+r}^{\overline{22}} dt + S\int_r^\infty e^{-\delta(t-r)}\, {}_{t-r} p_{x+r}^{\overline{22}}\, \mu_{x+r+t}^{23}\, dt$$

where the first integral gives the EPV of the disability annuity, and the second integral gives the EPV of the benefit on death.

Also

$$\begin{aligned} V_1(r,\infty) \quad = \quad & -P\int_r^\infty e^{-\delta(t-r)}\, {}_{t-r} p_{x+r}^{\overline{11}} dt + S\int_r^\infty e^{-\delta(t-r)}\, {}_{t-r} p_{x+r}^{\overline{11}}\, \mu_{x+r+t}^{13}\, dt \\ & + \int_r^\infty e^{-\delta(t-r)}\, {}_{t-r} p_{x+r}^{\overline{11}}\, \mu_{x+r+t}^{12}\, V_2(t,\infty) dt \end{aligned}$$

where the first integral gives the EPV of premium income, the second integral gives the EPV of the death benefit should death occur from State 1, and the third integral gives the EPV of benefits payable on transition to State 2.

Note that in each of Examples 4.1 and 4.2, we can essentially use a traditional actuarial approach to evaluate $V_i(r,u)$.

Example 4.3: *In the DII model, suppose that net premiums are payable continuously at rate P per annum while a life is in the healthy state, an annuity is payable to the policyholder at rate A per annum while disabled, and there is no death benefit. We have*

$$V_1(r,\infty) = -P\int_r^{\infty} e^{-\delta(t-r)} \; {}_{t-r}p_{x+r}^{\overline{11}} dt + \int_r^{\infty} e^{-\delta(t-r)} \; {}_{t-r}p_{x+r}^{\overline{11}} \mu_{x+t}^{12} V_2(t,\infty) dt \qquad (4.1)$$

and

$$V_2(r,\infty) = A\int_r^{\infty} e^{-\delta(t-r)} \; {}_{t-r}p_{x+r}^{\overline{22}} \, dt + \int_r^{\infty} e^{-\delta(t-r)} \; {}_{t-r}p_{x+r}^{\overline{22}} \mu_{x+t}^{21} V_1(t,\infty) dt. \qquad (4.2)$$

Note that in the second integral in equation (4.1), the term $V_2(t,\infty)$ takes account of all payments after time t whether the life is in State 1 or State 2, and similarly in (4.2).

The key difference between Example 4.3 and Examples 4.1 and 4.2 is that we cannot apply standard actuarial techniques to find $V_i(r,\infty)$.

Let us now consider equation (4.1). Writing

$$\mu_x^{12} + \mu_x^{13} = \mu_x^1$$

as the total transition intensity out of State 1 at age x, note that

$${}_{t-r}p_{x+r}^{\overline{11}} = \exp\left\{-\int_{x+r}^{x+t} \mu_s^1 ds\right\} = \exp\left\{-\int_0^{x+t} \mu_s^1 ds\right\} \div \exp\left\{-\int_0^{x+r} \mu_s^1 ds\right\}.$$

Thus equation (4.1) becomes

$$\begin{aligned} &\exp\left\{-\int_0^{x+r} \mu_s^1 ds\right\} V_1(r,\infty) \\ &= e^{\delta r}\Big(-P\int_r^{\infty} e^{-\delta t} \exp\left\{-\int_0^{x+t} \mu_s^1 ds\right\} dt \\ &+ \int_r^{\infty} e^{-\delta t} \exp\left\{-\int_0^{x+t} \mu_s^1 ds\right\} \mu_{x+t}^{12} V_2(t,\infty) dt\Big) \end{aligned}$$

and differentiation gives

$$- \mu^1_{x+r} \exp\left\{-\int_0^{x+r} \mu^1_s ds\right\} V_1(r,\infty) + \exp\left\{-\int_0^{x+r} \mu^1_s ds\right\} \frac{d}{dr} V_1(r,\infty)$$

$$= \delta \exp\left\{-\int_0^{x+r} \mu^1_s ds\right\} V_1(r,\infty) + P \exp\left\{-\int_0^{x+r} \mu^1_s ds\right\}$$

$$- \exp\left\{-\int_0^{x+r} \mu^1_s ds\right\} \mu^{12}_{x+r} V_2(r,\infty)$$

or

$$\frac{d}{dr} V_1(r,\infty) = \delta V_1(r,\infty) + P - \mu^{12}_{x+r}\left(V_2(r,\infty) - V_1(r,\infty)\right) + \mu^{13}_{x+r} V_1(r,\infty). \quad (4.3)$$

Similarly

$$\frac{d}{dr} V_2(r,\infty) = \delta V_2(r,\infty) - A - \mu^{21}_{x+r}\left(V_1(r,\infty) - V_2(r,\infty)\right) + \mu^{23}_{x+r} V_2(r,\infty). \quad (4.4)$$

We remark that equation (4.3) has a natural interpretation. The left hand side gives the rate of change of the reserve. On the right hand side, the reserve changes as follows: it is increased by interest and premium income and by the release of reserve on death, and is reduced by the difference between the reserves in States 1 and 2 when a transfer to State 2 occurs.

Equations (4.3) and (4.4), together with boundary conditions, are all we need to solve for $V_1(r,\infty)$ and $V_2(r,\infty)$ given the transition intensities.

In the general setting of n states, suppose that $B_{jk}(t)$ denotes the benefit payable at time t on transition from State j to State k, and $b_j(t)$ denotes the rate of benefit payment (positive or negative) at time t if the policyholder is in State j. Then

$$V_i(r,u) = \int_r^u e^{-\delta(t-r)} \, {}_{t-r}p^{\overline{ii}}_{x+r} \, b_i(t) dt$$
$$+ \int_r^u e^{-\delta(t-r)} \, {}_{t-r}p^{\overline{ii}}_{x+r} \sum_{j \neq i} \mu^{ij}_{x+t}\left(B_{ij}(t) + V_j(t,u)\right) dt$$

which leads to

$$\frac{d}{dr} V_i(r,u) = \delta V_i(r,u) - b_i(r) - \sum_{j \neq i} \mu^{ij}_{x+r}\left(B_{ij}(r) + V_j(r,u) - V_i(r,u)\right) \quad (4.5)$$

for $i = 1,2,...,n$. Equations (4.5) are known as generalisations of Thiele's differential equation. The basic Thiele's differential equation applies to the two-state "alive-dead" model. For example, for a whole life policy issued to a life aged x with sum assured S payable immediately on death and with annual net premiums of P payable continuously, Thiele's differential equation is

$$\frac{d}{dr}V_1(r,\infty) = \delta V_1(r,\infty) + P - \mu_{x+r}^{12}\left[S - V_1(r,\infty)\right].$$

4.2 Numerical implementation

Consider the DII model with net premiums payable continuously at rate P per annum while a life is in the healthy state, an annuity payable to the policyholder at rate A per annum while disabled, and a death benefit of S payable on death from either live state. To calculate P we find the EPV at time 0 of a unit payment while the life is in each of State 1 and State 2, and of a benefit of 1 on transfer to the dead state. Assume the policy term is u (years).

First, we consider, in our general notation, $b_1(t) = 1$, $b_2(t) = b_3(t) = 0$ and $B_{ij}(t) = 0$ for all i and j. Then equations (4.5) give

$$\frac{d}{dr}V_1(r,u) = \delta V_1(r,u) - 1 - \mu_{x+r}^{12}\left(V_2(r,u) - V_1(r,u)\right) + \mu_{x+r}^{13}V_1(r,u) \quad (4.6)$$

and

$$\frac{d}{dr}V_2(r,u) = \delta V_2(r,u) - \mu_{x+r}^{21}\left(V_1(r,u) - V_2(r,u)\right) + \mu_{x+r}^{23}V_2(r,u). \quad (4.7)$$

Solving for $V_1(0,u)$ gives us the EPV of a unit payment to the life while the life is in State 1, say α.

Second, we consider $b_1(t) = 0$, $b_2(t) = 1$, $b_3(t) = 0$ and $B_{ij}(t) = 0$ for all i and j. Then equations (4.5) give

$$\frac{d}{dr}V_1(r,u) = \delta V_1(r,u) - \mu_{x+r}^{12}\left(V_2(r,u) - V_1(r,u)\right) + \mu_{x+r}^{13}V_1(r,u) \quad (4.8)$$

and

$$\frac{d}{dr}V_2(r,u) = \delta V_2(r,u) - 1 - \mu_{x+r}^{21}\left(V_1(r,u) - V_2(r,u)\right) + \mu_{x+r}^{23}V_2(r,u). \quad (4.9)$$

Solving for $V_1(0,u)$ gives us the EPV of a unit payment to the life while the life is in State 2, say β.

Third, we consider $b_1(t) = b_2(t) = b_3(t) = 0$ and $B_{j3}(t) = 1$ for $j = 1,2$. Then equations (4.5) give

$$\frac{d}{dr}V_1(r,u) = \delta V_1(r,u) - \mu_{x+r}^{12}\left(V_2(r,u) - V_1(r,u)\right) - \mu_{x+r}^{13}\left(1 - V_1(r,u)\right) \quad (4.10)$$

and

$$\frac{d}{dr}V_2(r,u) = \delta V_2(r,u) - \mu_{x+r}^{21}\left(V_1(r,u) - V_2(r,u)\right) - \mu_{x+r}^{23}\left(1 - V_2(r,u)\right). \quad (4.11)$$

Solving for $V_1(0,u)$ gives us the EPV of a unit payment to the life on transfer to State 3, say γ.

Given the benefit levels, A and S, setting

$$P\alpha = A\beta + S\gamma$$

gives the net annual premium P.

In the above, we have set out three pairs of equations. This allows us to value income and each benefit. We could have set up two pairs of equations: one to value income, the other to value outgo.

Example 4.4 *Suppose that* $\delta = \log 1.045 = 0.044017$ *and, as in Example 3.2,*

$$\begin{aligned}
\mu_x^{13} &= \mu_x^{23} = 0.0005 + 0.000075858 \times 10^{0.038x}, \\
\mu_x^{12} &= 0.0004 + 0.0000034674 \times 10^{0.06x}, \\
\mu_x^{21} &= 0.005.
\end{aligned}$$

Consider the following insurance contract:

1. *Premiums are payable continuously while the life is in the healthy state at rate* P *per annum.*
2. *An annuity is payable to the policyholder at the rate of* $A = 50{,}000$ *per annum while the life is in the disabled state.*
3. *A death benefit of* $S = 100{,}000$ *is payable on transfer to the dead state from either live state.*
4. *The life is aged 30 at issue, and the term is 30 years.*

Setting $u = 30$*, we note that in solving each pair of differential equations, the boundary conditions are*

$$V_1(u,u) = 0 \quad \text{and} \quad V_2(u,u) = 0.$$

Having solved for P *as* $1{,}310.78$*, we then use*

$$\frac{d}{dr}V_1(r,u) = \delta V_1(r,u) + P - \mu_{x+r}^{12}\left(V_2(r,u) - V_1(r,u)\right) - \mu_{x+r}^{13}\left(S - V_1(r,u)\right) \quad (4.12)$$

and

$$\frac{d}{dr}V_2(r,u) = \delta V_2(r,u) - A - \mu_{x+r}^{21}\left(V_1(r,u) - V_2(r,u)\right) - \mu_{x+r}^{23}\left(S - V_2(r,u)\right) \quad (4.13)$$

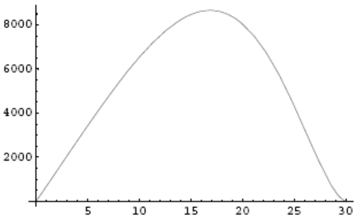
to find the functions $V_1(r,u)$ *and* $V_2(r,u)$ *for* $0 < r < u$*, where* $u = 30$. *The Mathematica commands to solve equations (4.6) to (4.13) are shown below.*

```
In[1]:= μ[x_] := 0.0005 + 0.000075858 * (10 ^ (0.038 * x))

In[2]:= ν[x_] := 0.0005 + 0.000075858 * (10 ^ (0.038 * x))

In[3]:= σ[x_] := 0.0004 + 0.0000034674 * (10 ^ (0.06 * x))

In[4]:= ρ = 0.005

Out[4]= 0.005

In[5]:= δ = Log[1.045]

Out[5]= 0.0440169
```

```
In[6]:= NDSolve[
      {V'[r] == δ * V[r] - 1 - σ[30 + r] * (W[r] - V[r]) +
          μ[30 + r] * V[r], W'[r] ==
         δ * W[r] - ρ * (V[r] - W[r]) + ν[30 + r] * W[r],
       V[30] == 0, W[30] == 0}, {V, W}, {r, 0, 30}]

Out[6]= {{V → InterpolatingFunction[{{0., 30.}}, <>],
       W → InterpolatingFunction[{{0., 30.}}, <>]}}

    (* V represents V1 (r,u),
     W represents V2 (r,u), output is
      stored in interpolating functions *)

    (* This solves Equations (4.6) and (4.7) *)

In[7]:= V[0] /. %6

Out[7]= {15.7628}

    (* V[0] is the EPV at force of
      interest 4.5 % pa of an annuity of
      1 pa payable continuously for 30
      years as long as (30) is in State 1 *)

In[8]:= Clear[V]

In[9]:= Clear[W]

    (* These commands delete the
      contents of the functions V and W *)

In[10]:= NDSolve[
      {V'[r] == δ * V[r] - σ[30 + r] * (W[r] - V[r]) +
          μ[30 + r] * V[r], W'[r] ==
         δ * W[r] - 1 - ρ * (V[r] - W[r]) + ν[30 + r] * W[r],
       V[30] == 0, W[30] == 0}, {V, W}, {r, 0, 30}]

Out[10]= {{V → InterpolatingFunction[{{0., 30.}}, <>],
        W → InterpolatingFunction[{{0., 30.}}, <>]}}

    (* This solves Equations (4.8) and (4.9) *)
```

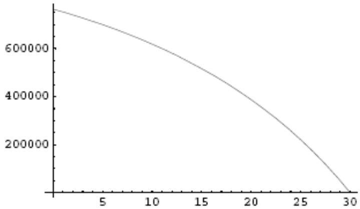
```
In[11]:= V[0] /. %10

Out[11]= {0.276552}

    (* V[0] is the EPV at force of
      interest 4.5 % pa of an annuity of
      1 pa payable continuously for 30
      years as long as (30) is in State 2 *)

In[12]:= Clear[V]

In[13]:= Clear[W]

In[14]:= NDSolve[
      {V'[r] == δ * V[r] - σ[30 + r] * (W[r] - V[r]) -
          μ[30 + r] * (1 - V[r]), W'[r] == δ * W[r] -
          ρ * (V[r] - W[r]) - ν[30 + r] * (1 - W[r]),
       V[30] == 0, W[30] == 0}, {V, W}, {r, 0, 30}]

Out[14]= {{V → InterpolatingFunction[{{0., 30.}}, <>],
       W → InterpolatingFunction[{{0., 30.}}, <>]}

    (* This solves Equations (4.10) and (4.11) *

In[15]:= V[0] /. %14

Out[15]= {0.0683401}

    (* V[0] is the EPV at force of interest
      4.5 % pa of 1 payable on transition to
      State 3 from either State 1 or State 2 *)

In[16]:= P = (50000 * 0.276551867973955 +
          100000 * 0.06834010453363834) /
       15.762797577927241

Out[16]= 1310.78

    (* P is the total premium payable pa *)

In[17]:= Clear[V]

In[18]:= Clear[W]

    (* We can calculate the EPV of all benefits
      directly by solving the following *)
```

```
In[19]:= NDSolve[
    {V'[r] == δ * V[r] - σ[30 + r] * (W[r] - V[r]) -
        μ[30 + r] * (100000 - V[r]),
     W'[r] == δ * W[r] - 50000 - ρ * (V[r] - W[r]) -
        ν[30 + r] * (100000 - W[r]),
     V[30] == 0, W[30] == 0}, {V, W}, {r, 0, 30}]

Out[19]= {{V → InterpolatingFunction[{{0., 30.}}, <>],
    W → InterpolatingFunction[{{0., 30.}}, <>]}}

In[20]:= V[0] /. %19

Out[20]= {20661.6}

  (* Set up the solution of Equations (4.12)
    and (4.13) for net premium reserves *)

In[21]:= NDSolve[
    {V'[r] == δ * V[r] + P - σ[30 + r] * (W[r] - V[r]) -
        μ[30 + r] * (100000 - V[r]),
     W'[r] == δ * W[r] - 50000 - ρ * (V[r] - W[r]) -
        ν[30 + r] * (100000 - W[r]),
     V[30] == 0, W[30] == 0}, {V, W}, {r, 0, 30}]

Out[21]= {{V → InterpolatingFunction[{{0., 30.}}, <>],
    W → InterpolatingFunction[{{0., 30.}}, <>]}}

In[22]:= Plot[V[r] /. %21, {r, 0, 30}]
```




```
Out[22]= - Graphics -
```

```
In[23]:= Plot[W[r] /. %21, {r, 0, 30}]
```




```
Out[23]= - Graphics -

In[24]:= TableForm[
          Table[{r, Round[V[r] /. %21]}, {r, 0, 30, 5}]]

Out[24]//TableForm=
     0      0
     5      3431
     10     6532
     15     8468
     20     8044
     25     4343
     30     0

     (* Answers are rounded
       to the nearer integer *)
```

5 Further reading

The original important contribution on generalisations of Thiele's differential equation is by Hoem (1969). Extensions can be found in Hoem (1988) and Wolthuis (1995). See also Norberg (1995) who considers higher moments of present values. A full discussion of the DII model is given by Waters (1984).

A standard approach to solving differential equations numerically is the Runge-Kutta method. Details can be found in texts such as Conte and De Boor (1980).

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