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Prepared by Jonathan Ziveyi, Michael Sherris, Yang Shen

Presented to the Actuaries Institute
ASTIN, AFIR/ERM and IACA Colloquia
23-27 August 2015
Sydney

Institute of Actuaries of Australia
ABN 69 000 423 656
Level 2, 50 Carrington Street, Sydney NSW Australia 2000
t +61 (0) 2 9233 3466 **f** +61 (0) 2 9233 3446
e actuaries@actuaries.asn.au **w** www.actuaries.asn.au

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Yang Shen,[*] Michael Sherris[†] and Jonathan Ziveyi[‡]

May 13, 2015

Abstract

We consider the pricing of guaranteed minimum maturity benefits (GMMB) embedded in variable annuity contracts in the case where the guarantees can be surrendered at any time prior to maturity. Surrender charges are imposed as a way of discouraging early termination of variable annuity contracts. We formulate the problem as an American put option pricing and derive the corresponding pricing partial differential equation (PDE) using hedging arguments and Itô's Lemma. Given the underlying stochastic evolution of the fund, we also present the associated transition density PDE whose solution is well known in the literature. An explicit integral expression for the pricing PDE is then presented with the aid of Duhamel's principle. An expression for the sensitivity of the guarantee fees with respect to changes in the underlying fund value (called the "delta") is also derived; the delta can be used for risk management purposes. We then outline the algorithm for implementing the integral expressions for the price, the corresponding early exercise boundary and the delta of the surrender option. We wrap up the paper by presenting numerical results and analyzing the sensitivity of the prices, early exercise boundaries and deltas to changes in the underlying variables.

JEL Classification

Keywords:

[*]yang.shen1@unsw.edu.au; CEPAR and School of Risk and Actuarial Studies, UNSW, Sydney, NSW 2052, Australia.

[†]m.sherris@unsw.edu.au; CEPAR and School of Risk and Actuarial Studies, UNSW, Sydney, NSW 2052, Australia.

[‡]Corresponding Author: j.ziveyi@unsw.edu.au; School of Risk and Actuarial Studies, UNSW, Sydney, NSW 2052, Australia.

1 Introduction

A variable annuity is an insurance contract where a policyholder pays either a single premium or a stream of periodic payments during the accumulation phase in return of minimum guaranteed payments from the insurer during the annuitization phase. The guarantees embedded in variable annuity contracts protect policyholders from unanticipated market movements. These guarantees exhibit financial option-like features, naturally leading to the way they are valued in practise. There are two major classes of guarantees; guaranteed minimum death benefits (GMDBs) and the guaranteed minimum living benefits (GMLBs).

GMDBs are usually offered during the accumulation phase and providing guaranteed payments of the accumulated value of premiums to beneficiaries in the event of untimely death of the policyholder during the accumulation phase. GMLBs provide principal and/or income guarantees to help protect the policyholder's income from declining during the accumulation phase. GMLBs can further be categorised into three subclasses; namely, GMxB, where "x" stands for maturity (M), income (I) and withdrawal (W). A GMMB guarantees the return of the premium payments made by the policyholder or a higher stepped-up value at the end of the accumulation period, while a GMIB guarantees a lifetime income stream when a policyholder annuitizes the GMMB regardless of the underlying investment performance. A GMWB guarantees withdrawal of a stream of payments to policyholder regardless of the contract account value.

Insurance companies usually charge proportional fees on variable annuity accounts in funding the guarantees. If such fees are too high relative to the performance of the fund, the policyholder can choose to surrender the contract or the guarantee prior to maturity in return of a surrender benefit. The benefits are usually net of surrender/penalty charged enforced as way of discouraging early termination of the contracts. In this paper we will focus on the valuation of the GMMB rider embedded in a variable annuity contract in the case where the guarantee can be surrendered anytime prior to maturity. This valuation problem has received little attention in the literature. Shen and Xu (2005) consider the fair valuation of equity-linked policies with interest rate guarantees in the presence of surrender options using partial differential equation approach. The valuation problem is reduced to a free boundary problem which can be solved using a variety of numerical techniques such as finite difference schemes. The authors also derive explicit Black and Scholes (1973) type solutions for the case where there are no surrender options. Constabile et al. (2008) consider a similar valuation problem and devise the binomial trees approach to generate fair premium values.

Bauer et al. (2008) provide a general framework for consistent pricing of various types of guarantees embedded in variable annuities being traded in the market. The authors present an extensive analysis of the guarantees by incorporating the possibility of surrendering the contracts anytime prior to maturity. Bacinello (2013) considers the pricing of participating life insurance policies with surrender options using recursive binomial trees approach.

In this paper we provide alternative derivations and representation of the GMMB embedded in a variable annuity contract where the policyholder can surrender the guarantee anytime prior to maturity. Our main result involves the decomposition of the variable annuity contract into a mutual fund and a GMMB that can be surrendered any time prior to maturity. The surrender option feature on the GMMB makes valuation problem similar to that encountered in American put option pricing. We then focus on valuing the surrender option which we present as an optimal stopping problem. Using techniques developed in Jacka(1991), we then transform the stopping time problem into a free-boundary problem leading to an equivalent representation in Shen and Xu (2005). By using the techniques developed in Jamshidian (1992) for transforming the free-boundary problem to a non-homogeneous partial differential equation (PDE), we derive the general solution of the PDE in integral with the aide of Duhamel's principle. Expressions for the corresponding early exercise boundary and the delta which is the sensitivity of the option price with respect to changes in the fund value are also derived.

Numerical results quantifying the early exercise boundary profiles, premiums to be charged per guarantee and the corresponding delta profiles are also provided. Our main finding is that when surrender charges are relatively high, it is advisable to delay exercising the guarantee early as significant amount of surrender benefits may end up being consumed by early termination charges. We perform numerical comparisons between standard American put options and surrender options by assessing the impact of continuously compounded insurance charges surrender fees on the premium to be levied on guarantees. We note that premium values for surrender options are consistently higher than the corresponding American put option prices.

In this paper, we purely focus on valuing the guarantee component of the variable annuity contract and use Jamshidian (1992) approach differentiating it from the work of Bernard et al.

(2014) who use techniques developed in Carr et al. (1992) to derive the representation of the optimal surrender strategy for a variable annuity contract embedded with guaranteed minimum accumulation benefits(GMAB). In deriving the pricing framework, the authors treat the entire variable annuity contract (the mutual fund plus the GMAB) as a single underlying asset and then derive the corresponding pricing formulas.

The rest of the paper is structured as follows. Section 2 sets up the model dynamics and relates the valuation of GMMB with an American option pricing problem. The general integral solution of the valuation problem is presented in Section 3 together with expressions for the corresponding early exercise boundary and delta of the option. Algorithms for numerically implementing the valuation expressions are presented in Section 4. Numerical results are then presented in Section 5 followed by concluding remarks in Section 6. Lengthy derivations and proofs have been relegated to the Appendices.

2 Problem Statement

Let $[0, T]$ be a finite horizon and $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space carrying a one-dimensional standard Brownian motion $W_t = (W_t)_{0 \leq t \leq T}$. Here $\mathbb{Q}$ is the risk-neutral probability measure. We denote by $\mathbb{E}[\cdot]$ the expectation under $\mathbb{Q}$. Let $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ be a filtration generated by W_t satisfying the usual conditions of right-continuity and $\mathbb{P}$-completeness. We consider a variable annuity contract embedded with a guaranteed minimum maturity benefit (GMMB) rider where the policyholder can choose to surrender the guarantee anytime prior to maturity. The policyholder's premium will be invested in a fund consisting of units of an underlying asset, $S = (S_t)_{0 \leq t \leq T}$, whose risk-neutral evolution is governed by the geometric Brownian motion process

$$dS_t = rS_t dt + \sigma S_t dW_t, \tag{1}$$

where $r > 0$ and $\sigma > 0$ are the risk-free interest rate and the volatility of the underlying asset, respectively. The fund value at time t is denoted as

$$F_t = e^{-ct} S_t, \tag{2}$$

where c is the continuously compounded insurance charge levied on the fund value by the variable annuity provider (see Milevsky and Salisbury (2001)). By applying Itô's Lemma it can be shown that the risk-neutral dynamics of the fund value, $F = (F_t)_{0 \leq t \leq T}$, satisfies

$$dF_t = (r - c)F_t dt + \sigma F_t dW_t. \tag{3}$$

Using risk-neutral arguments, the fund value at initial time net of initial expense charges can be represented as the discounted expected value of the terminal payout, that is

$$\begin{aligned} F_0 &= \mathbb{E}\left[e^{-rT} \max(G_T, F_T)\right] \\ &= \mathbb{E}\left[e^{-rT} F_T\right] + \mathbb{E}\left[e^{-rT} \max(G_T - F_T, 0)\right], \end{aligned} \tag{4}$$

where F_T is the fund value at maturity time, T, given that the fee charged during $t \in [0, T]$ is equal to c, and G_T is the guaranteed value at maturity of the contract. Equation (4) is made up of two components: the first being the discounted expected value of the terminal fund value and the second being a put option which is equivalent to a guarantee rider to be exercised only if the terminal fund value is below the guaranteed amount, G_T.

Whilst equation (4) is akin to a standard European put option, various events may necessitate early termination of the guarantee rider. Among such events include; the guarantee being deep out-of-the-money, in such a case, the policyholder will find it cheaper not to hold the guarantee as the probability of it ending up in-the-money will be very low. The second case is when continuation value of the guarantee is equal to the immediate exercise value prior to maturity. In the event of the guarantee being terminated prior to maturity, early termination/ surrender charges will apply such that the resulting the benefit will be

$$(1 - \kappa_t)F_t, \tag{5}$$

with κ_t being the penalty percentage charged for surrendering at time t. Milevsky and Salisbury (2001) interpret κ_t as an incentive to remain in the variable annuity contract and as mechanism for funding the guarantee. As in Bernard et al. (2014), we assume that κ_t is exponentially decreasing and is equal to $1 - e^{-\kappa(T-t)}$ implying that the surrender benefit at anytime $t \in [0, T]$ is equal to

$$e^{-\kappa(T-t)} F_t. \tag{6}$$

For optimality in exercising the guarantee (surrender option) early, we will assume that the inequality $\kappa < c < r$ holds otherwise the option will be held to maturity. As will be explained below, this inequality insures that the surrender charges will not exceed/erode the benefits of exercising the option early. From equation (4), the variable annuity account at maturity can be represented as

$$F_T + \max(G_T - F_T, 0), \tag{7}$$

and anytime prior to maturity[1], this can be represented as

$$F_t = e^{-r(T-t)}\mathbb{E}\left[F_T\right] + \operatorname*{ess\,sup}_{t\le\tau^*\le T} e^{-r(\tau^*-t)}\mathbb{E}\left[\max(G_T - e^{-\kappa(T-\tau^*)}F_{\tau^*}, 0)|\mathcal{F}_t\right], \tag{8}$$

where the supremum is taken over all stopping times, τ^*. The first component on the right-hand side of equation (8) is the discounted expectation of maturity value of the fund which can evaluated easily using a variety of techniques such as the equilibrium asset pricing theory. The second component is a typical American put option which we reproduce here as

$$C(t,F) = \operatorname*{ess\,sup}_{t\le\tau^*\le T} e^{-r(\tau^*-t)}\mathbb{E}\left[\max(G_T - e^{-\kappa(T-\tau^*)}F_{\tau^*}, 0)|\mathcal{F}_t\right]. \tag{9}$$

The valuation problem in equation (9) is an optimal stopping time problem.

Proposition 2.1. *It has been established that the fair value of the American put option in* (9) *is a unique strong solution of the free boundary problem*[2]

$$\frac{\partial C}{\partial t} + (r-c)F\frac{\partial C}{\partial F} + \frac{1}{2}\sigma^2F^2\frac{\partial^2 C}{\partial F^2} - rC = 0, \tag{10}$$

where $B_t < e^{-\kappa(T-t)}F < \infty$ *with* B_t *being the optimal early exercise boundary below which the put option will be exercised. The partial differential equation (PDE)* (10) *is solved subject to the boundary and terminal conditions*

$$C(T,F) = \max(G_T - F, 0), \tag{11}$$

$$\lim_{F\to\infty} C(t,F) = 0, \quad t\in[0,T], \tag{12}$$

$$C(t,B_t) = G_T - B_t, \quad t\in[0,T], \tag{13}$$

$$\lim_{F\to B_t e^{\kappa(T-t)}} \frac{\partial C}{\partial F} = -e^{-\kappa(T-t)}, \quad t\in[0,T]. \tag{14}$$

Proof. Refer to the proof of Proposition 2.7 in Jacka (1991). Also, the PDE can be derived by applying standard hedging arguments and Itô's Lemma to a portfolio consisting of a put option, $C(t,F)$, and optimal units of the underlying fund, F, using the arguments presented in Black and Scholes (1973) where dynamics of F is governed by the SDE presented in equation (3). □

[1]Due to the possibility of surrendering the contract early.

[2]See Jacka (1991).

The underlying asset domain for the PDE (10) is bounded below by the early exercise boundary, B_t. Jamshidian (1992) shows that one can consider an unbounded domain for the underlying asset by noting that at anytime, $t \in [0, T]$, below the early exercise boundary

$$C(t, F) = G_T - e^{-\kappa(T-t)} F_t, \tag{15}$$

implying that the following equation holds

$$\frac{\partial C}{\partial t} + (r - c)F\frac{\partial C}{\partial F} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 C}{\partial F^2} - rC = (c - k)e^{-\kappa(T-t)}F - rG_T. \tag{16}$$

Above the early exercise boundary, the option will be held and satisfies equation (10). Combining equations (10) and (16), and using the fact that $C(t, F)$ is a continuous differentiable function in F at B_t yield

$$\frac{\partial C}{\partial t} + (r - c)F\frac{\partial C}{\partial F} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 C}{\partial F^2} - rC + \mathbb{1}_{\{F \leq B_t e^{\kappa(T-t)}\}} \left[rG_T - (c - k)e^{-\kappa(T-t)}F \right] = 0, \tag{17}$$

where $\mathbb{1}_{\{x \leq B\}}$ is an indicator function which is equal to one if $x \leq B$ or zero otherwise. Equation (17) is defined on the domain $0 < F < \infty$ and this motivates the solution approach we present below.

We now explain the economic intuition of the inhomogenous term in equation (17). Suppose that at time t, $e^{-\kappa(T-t)}F_t < G$ implying that it is optimal to exercise the option. If the option is not exercised now, it can still be exercised at the next instant $t + dt$ because the fund value function is a continuous-time process. All the policyholder will lose by not exercising now is the instantaneous interest $rGdt$, but would save on the early termination charges $(c - \kappa)e^{-\kappa[T-(t+dt)]}F_{t+dt}dt$ such that the total net loss would be

$$[rG - (c - \kappa)e^{-\kappa[T-(t+dt)]}F_{t+dt}]dt. \tag{18}$$

However if the variable annuity provider were to compensate the policyholder with an equivalent amount, then the policyholder will be indifferent to delaying exercise to the next instant. Suppose that the two counterparties agree to prohibit exercise until maturity, and in return, the policyholder is continuously compensated by the amount equivalent to equation (18) for delaying optimal exercise when it is optimal to do so. Whenever $e^{-\kappa(T-t)}F_t$ is above the early exercise boundary, the compensation will be zero since exercising is not optimal. This leads to the conclusion that the guaranteed minimum maturity benefit embedded in a variable annuity with a surrender option is a typical American put option consisting of a European option component plus a contract that pays a continuous cashflow presented in equation (18). An equivalent quantity to the continous cashflow in Jamshidian (1992) is termed the "cost-of-carry" of an option, which is compensation for delayed exercise.

Also associated with the SDE (3) is the corresponding transition density function denoted here[3] as $H(\tau, F; F_0)$. This function represents the probability of passage from state F at time-to-maturity, τ, to state F_0 at maturity of the variable annuity contract. The transition density function satisfies the backward Kolmogorov PDE

$$\frac{\partial H}{\partial \tau} = (r-c)F\frac{\partial H}{\partial F} + \frac{1}{2}\sigma^2 F^2 \frac{\partial^2 H}{\partial F^2}, \tag{19}$$

where $0 \le F < \infty$. Equation (19) is solved subject to the terminal condition

$$H(0, F; F_0) = \delta(F - F_0), \tag{20}$$

where $\delta(\cdot)$ is a Dirac delta function.

Now let $F = e^x$ and $C(\tau, e^x) \equiv V(\tau, x)$ such that the PDE in (17) transforms to

$$\frac{\partial V}{\partial \tau} = \phi\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} - rV + \mathbb{1}_{\{x \le \ln B + \kappa\tau\}}\left[rG - (c-k)e^{-\kappa\tau}e^x\right], \tag{21}$$

where $\phi = (r - c - \frac{1}{2}\sigma^2)$.

Applying similar transformations to the transition density PDE and letting $H(\tau, e^x) \equiv U(\tau, x)$ equation (19) becomes

$$\frac{\partial U}{\partial \tau} = \phi\frac{\partial U}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 U}{\partial x^2}. \tag{22}$$

Equation (22) is solved subject to the terminal condition

$$U(0, x; x_0) = \delta(x - x_0). \tag{23}$$

3 Main Results

Having outlined the problem statement in Section 2, we now present the integral representations of premium values for surrender options together with the corresponding early exercise boundary

[3]In what follows, we let $\tau = T - t$ where τ is the time-to-maturity, and where appropriate we suppress the dependence of the fund value on t.

which needs to be determined as part of the solution. We also present the delta expression for the surrender option which quantifies the sensitivity of the premium values to changes in the underlying fund.

Proposition 3.1. *By using Duhamel's principle, the general solution of equation* (21) *can be represented as*

$$V(\tau, x) = V_E(\tau, x) + V_P(\tau, x), \tag{24}$$

where

$$V_E(\tau, x) = e^{-r\tau} \int_{-\infty}^{\infty} (G - e^w)^+ U(\tau, x; w) dw, \tag{25}$$

and

$$V_P(\tau, x) = \int_0^\tau e^{-r(\tau-\xi)} \int_{-\infty}^{\ln B_\xi + \kappa(\tau-\xi)} [rG - (c-k)e^{-\kappa(\tau-\xi)} e^w] U(\tau - \xi, x; w) dw d\xi. \tag{26}$$

The first term on the right hand side of equation (24) *is the European option component and the second term is the early exercise premium component. The function, $U(\tau, x; w)$ is the univariate normal transition density function, i.e.,*

$$U(\tau, x; w) = \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left\{ -\frac{(x - w + \phi\tau)^2}{2\tau\sigma^2} \right\}, \tag{27}$$

which is a solution to the transition density function PDE in (22).

Proof. The proof proceeds by substituting equation (24) into the the PDE (21) and then proceeding as detailed in Appendix B of Chiarella and Ziveyi (2014). □

Proposition 3.2. *The explicit solution of equation* (24) *can be represented as*

$$V(\tau, x) = V_E(\tau, x) + V_P(\tau, x), \tag{28}$$

where

$$V_E(\tau, x) = Ge^{-r\tau} \mathcal{N}(-d_2(\tau, x, G)) - e^x e^{-c\tau} \mathcal{N}(-d_1(\tau, x, G)), \tag{29}$$

and

$$\begin{aligned} V_P(\tau, x) = rG &\int_0^\tau e^{-r(\tau-\xi)} \mathcal{N}\left(-d_2\left(\tau - \xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right) d\xi \\ &- (c - \kappa)e^x \int_0^\tau e^{-(c+\kappa)(\tau-\xi)} \mathcal{N}\left(-d_1\left(\tau - \xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right) d\xi, \end{aligned} \tag{30}$$

with $\mathcal{N}(d)$ being a cumulative normal distribution function and

$$d_1(\tau, x, G) = \frac{x - \ln G + (r - c + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}, \quad d_2(\tau, x, G) = d_1(\tau, x, G) - \sigma\sqrt{\tau}. \tag{31}$$

Proof. Refer to Appendix A.1. □

The early exercise premium component in equation (30) is implicitly dependent on the early exercise boundary, B_τ, and hence needs to be determined as part of the solution. By using the value-matching condition presented in equation (13), the early exercise boundary is the solution to the implicit Volterra integral equation

$$\begin{aligned} G - B_\tau = {} & G e^{-r\tau}\mathcal{N}(-d_2(\tau, \ln B_\tau + \kappa\tau, G)) - B_\tau e^{-(c-\kappa)\tau}\mathcal{N}(-d_1(\tau, \ln B_\tau + \kappa\tau, G)) \\ & + rG\int_0^\tau e^{-r(\tau-\xi)}\mathcal{N}\left(-d_2\left(\tau - \xi, \ln B_\tau + \kappa\tau, B_\xi e^{\kappa(\tau-\xi)}\right)\right) d\xi \\ & - (c-\kappa)B_\tau e^{\kappa\tau}\int_0^\tau e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}\left(-d_1\left(\tau - \xi, \ln B_\tau + \kappa\tau, B_\xi e^{\kappa(\tau-\xi)}\right)\right) d\xi. \end{aligned} \tag{32}$$

Proposition 3.3. *The exercise boundary at maturity is*

$$B_0 = \min\left(1, \frac{r}{c-\kappa}\right) G. \tag{33}$$

Proof. Refer to Appendix A.2 □

From our earlier assumption that $\kappa < c < r$, it turns out that

$$B_0 = G, \tag{34}$$

which is the guaranteed fund value at maturity. At every other instant prior to maturity the early exercise boundary is determined by solving equation (32) recursively.

The sensitivity of the guarantee premium values to changes in the underlying factors is crucial when trading variable annuities. In option pricing, a family of such sensitivities is termed "Greeks" named after the Greek letters used to denote them. The most popular Greek in option pricing is the delta, which measures the degree to which an option is exposed to shifts in the price of the underlying asset. In our current situation, we will interpret the delta as the sensitivity of the guarantee premium to changes in the fund value. A negative delta implies that the guarantee fee decreases for every dollar increase in the fund value, so one is effectively short the fund through a long put option position. Delta works best for short-term options and does not tell the probability of how often the fund will hit the early exercise boundary between now and expiration, but merely the probability that it expires in the money. We present the delta expression in the next proposition.

Proposition 3.4. *By differentiting equation* (28) *with respect to the underlying fund value, the delta of the surrender option can be represented as*

$$D(\tau,x) = D_E(\tau,x) + D_P(\tau,x), \tag{35}$$

where

$$D_E(\tau,x) = -e^{-c\tau}\mathcal{N}(-d_1(\tau,x,G)), \tag{36}$$

and

$$\begin{aligned} D_P(\tau,x) = & -\frac{rG}{\sigma e^x}\int_0^\tau e^{-r(\tau-\xi)} n\left(-d_2\left(\tau-\xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{1}{\sqrt{\tau-\xi}}d\xi \\ & -(c-\kappa)\int_0^\tau e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}\left(-d_1\left(\tau-\xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right)d\xi \\ & +\frac{c-\kappa}{\sigma}\int_0^\tau e^{-(c+\kappa)(\tau-\xi)} n\left(-d_1\left(\tau-\xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{1}{\sqrt{\tau-\xi}}d\xi. \end{aligned} \tag{37}$$

with $n(d)$ being a density function of the standard normal distribution.

Proof. Refer to Appendix A.3. □

4 Numerical Implementation

Having formulated the guarantee equations as presented in (29) and (30), together with the early exercise boundary equation in (32) and the corresponding delta in (35) , we now outline a numerical technique for solving this system of equations.

We adopt the numerical integration techniques developed in Huang et al. (1996) who implement the American put option pricing framework developed in Kim (1990). Similar techniques have also being employed in Chiarella and Ziveyi (2014) when pricing American spread call options where the dynamics of the underlying assets evolve under the influence of geometric Brownian motion processes.

The European option component $V_E(\tau, x)$ involves a cumulative normal distribution function which can be easily handled by a variety of in-built software applications. However, such software applications cannot be readily applied to the early exercise premium component $V_P(\tau, x)$ as this term involves the entire history of the early exercise boundary, B_τ, which needs to be iteratively solved at each instant.

The early-exercise premium component also involves an integral with respect to the running time-to-maturity, ξ, which also makes use of the entire history of the early exercise boundary at each point in time.

To implement equations (29), (30),(32) and (35), we first discretise the time domain, τ, into M equally spaced subintervals of length $h = T/M$ and apply the extended Simpson's rule.

The numerical algorithm is initiated at maturity, $\tau_0 = 0$ where the exercise boundary is equal to the guarantee value presented in equation (34). This serves as the starting value for tracking the early exercise boundary backwards in time. We denote the time-steps as $\tau_m = mh$, for $m = 1, 2, \cdots, M$. The disretised version of the variable annuity guarantee is then represented as

$$V(mh, x) = V_E(mh, x) + V_P(mh, x), \tag{38}$$

where

$$V_E(mh, x) = Ge^{-r(mh)}\mathcal{N}(-d_2(mh, x, G)) - e^x e^{-c(mh)}\mathcal{N}(-d_1(mh, x, G)), \tag{39}$$

and

$$\begin{aligned} V_P(mh, x) = hrG \sum_{j=0}^{m} e^{-r(m-j)h}\mathcal{N}\left(-d_2\left((m-j)h, x, B(mh)e^{\kappa(m-j)h}\right)\right) w_j \\ - h(c-\kappa)e^x \sum_{j=0}^{m} e^{-(c+\kappa)(m-j)h}\mathcal{N}\left(-d_1\left((m-j)h, x, B(mh)e^{\kappa(m-j)h}\right)\right) w_j. \end{aligned} \tag{40}$$

Here, w_j are the weights of Simpson's rule for integration in the ξ direction while h is the corresponding step size. At each time step, we need to implicitly determine the early exercise boundary $B(mh)$ which also depends on its entire history up to the current time step. Root finding techniques are employed to accomplish this task. The discretised version of the value-matching condition equation can be shown to be

$$B(mh) = G - V(mh, x), \tag{41}$$

where $V(mh, x)$ is presented in equation (38). Likewise, the discretised version of the delta presented in equation (35) can be represented as

$$D(mh, x) = D_E(mh, x) + D_P(mh, x), \tag{42}$$

where

$$D_E(mh, x) = -e^{-c(mh)}\mathcal{N}(-d_1(mh, x, G)), \tag{43}$$

and

$$\begin{aligned} D_P(mh, x) = &-\frac{hrG}{\sigma e^x}\sum_{j=0}^{m} e^{-r(m-j)h} n\left(-d_2\left((m-j)h, x, B(mh)e^{\kappa(m-j)h}\right)\right)\frac{1}{\sqrt{(m-j)h}}w_j \\ &- h(c-\kappa)\sum_{j=0}^{m} e^{-(c+\kappa)(m-j)h}\mathcal{N}\left(-d_1\left((m-j)h, x, B(mh)e^{\kappa(m-j)h}\right)\right)w_j \\ &+ \frac{h(c-\kappa)}{\sigma}\sum_{j=0}^{m} e^{-(c+\kappa)(m-j)h} n\left(-d_1\left((m-j)h, x, B(mh)e^{\kappa(m-j)h}\right)\right)\frac{1}{\sqrt{(m-j)h}}w_j. \end{aligned} \tag{44}$$

5 Numerical Results

We now present numerical results obtained from implementing the framework presented in Section 4. For all numerical experiments that follow, we use the set of parameters presented in Table 1 unless stated otherwise. As highlighted in Bernard et al. (2014), such sensitivity analysis helps in shading light on the properties of the early exercise boundary and the guarantee fees. Note that our parameters choice is arbitrary and the model works for any reasonable choice of parameter set. The time domain has been discretised into 100 time steps, implying that $h = 0.15$ years when $\tau = 15$.

Parameter	G	τ	σ	r	c	κ
Value	100	15	0.20	0.05	0.03	0.01

Table 1: Parameters for the Variable Annuity Rider

Figure 1 shows the impact of varying the surrender charge, κ, on the early exercise boundary. We note that the early exercise boundary increases as the level of κ increases. Increasing levels of κ result in higher guarantee fees making it prohibitively expensive to surrender the guarantee early as revealed in Table 2. From this table, we note that when $c = 3\%$ for instance, varying the surrender charges from $\kappa = 0$ to $\kappa = 3\%$ (last three columns of the table) results in a gradual increase of guarantee fees. Thus when surrender charges are relatively high, it is advisable to delay exercising the guarantee early as a significant amount of surrender benefits may end up being used to settle for the early termination charges.

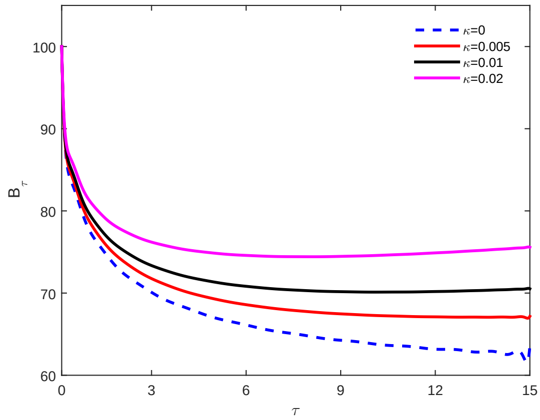
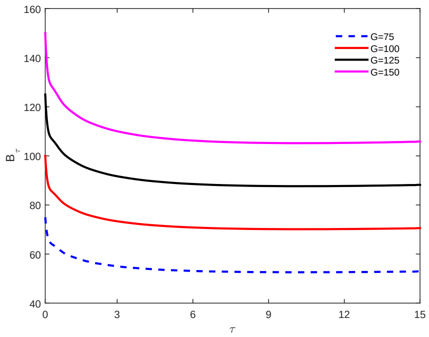


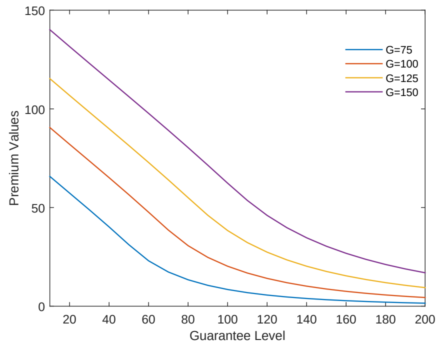
Figure 1: The impact of varying the surrender charges on the early exercise boundary.

We next assess the impact of varying the guarantee level on the early exercise boundary and the corresponding effects on the premiums to be charged for provision of such guarantees in Figure 2. In this figure, we vary the guarantee level and keep all other parameters constant as presented in Table 1. Increasing the guarantee levels result in higher exercise boundaries as revealed in Figure 2a, that is, when the guarantee level is increased, the corresponding exercise boundary is shifted upwards. As Figure 2b reveals, higher premiums must be levied for increasing levels of minimum guarantee. It naturally makes sense for insurers to charge higher premiums for increasing levels of minimum guarantees. Insurers can then use the premiums to devise appropriate hedging strategies to be used for generating the minimum guaranteed amounts in the event of such guarantees ending in-the-money or if the policyholder choose to surrender prior to maturity.

One of the most important concepts when trading options is volatility. Volatility measures how fast and how much prices of the underlying asset move. As such it is of great importance to understand how premiums to be levied on guarantees respond to changes in volatility. Figure 3 shows the effects of increasing volatility on the early exercise boundary and the premium values. We note from Figure 3a that as volatility increases, the corresponding early exercise boundary



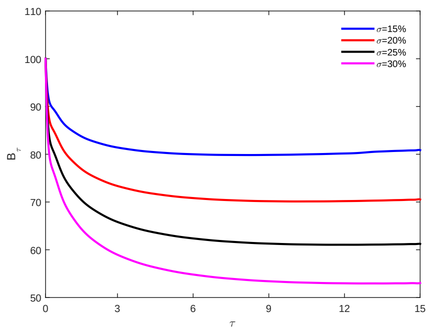
(a) Assessing how the early exercise boundary changes by varying the minimum guarantee.



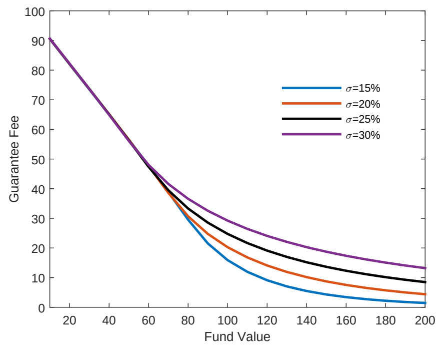
(b) Assessing the effects of vary the guarantee level on the premium values

Figure 2: Assessing the impacts of varying the guarantee levels on the early exercise boundary and premium values. All other parameters are as provided in Table 1

decreases. In option pricing theory, it also well established that an increase in volatility results in an increase in option prices. This is depicted in Figure 3b where premium values for near in-the-money and out-of-the-money guarantees increase with increasing levels of volatility. These findings are consistent with Bernard et al. (2014) who note that higher volatility causes an increase of the present value of the variable annuity liabilities.



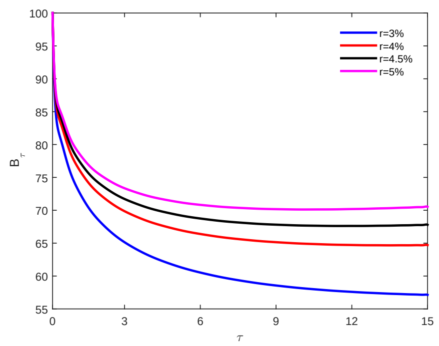
(a) The impact of varying the volatility level on the early exercise boundary.



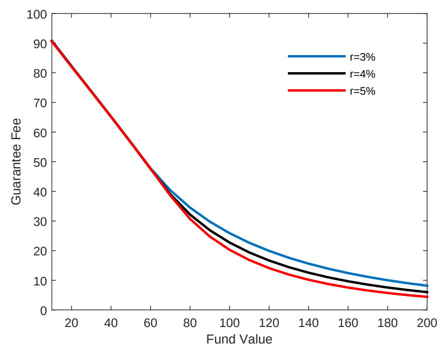
(b) Surrender option premiums for varying σ.

Figure 3: The effects of varying the volatility on the early exercise boundary and premium values. All other parameters are as presented in Table 1

It is also of interest to investigate the early exercise boundary and the corresponding guarantee premiums respond to changes in interest rates. From Figure 4a we note that as the level of interest rates is increased, early exercise boundary also increases. The rule of thumb when trading put options is that higher risk free interest rates mean cheaper put option prices, all things being equal. This is revealed in Figure 4b where we note a decrease in premiums as interest rates gradually increase from 3% to 5% for near at-the-money and out-of-the-money options. The interest rates are fully priced in for deep in-the-money guarantees hence the convergence of premiums as depicted in the figure.



(a) The impact of varying interest rates on the early exercise boundary.



(b) The impact of varying interest rates on guarantee premiums.

Figure 4: The effects of varying interest rates on the early exercise boundary and premium values. All other parameters are as presented in Table 1

It is also important to analyse premiums differences between surrender options and standard American put options. The formula for a standard put option on a non-dividend paying stock can be recovered by setting κ and c equal to zero in equation (28). In our analysis we subtract the implied standard American put option values from the associated guarantee values obtained by using the parameter set in Table 1. The standard American put option prices have been generated by implementing the algorithm devised in Kallast and Kivinukk (2003). The results of this analysis are presented in Figure 5. We note that near at-the-money (around the strike price which is the guarantee level) guarantee premiums are consistently higher than the corresponding standard American put option prices under the Black and Scholes (1973) framework, with the largest differences corresponding to at-the-money guarantees. Surrender options are more expensive to standard American put options and this reflects the effects of surrender charges and continuously compounded insurance charges levied on the fund value.

In Table 2 we further elaborate on how premium values change for various combinations of κ and *c*. As pointed above, we note that prices corresponding to the standard American put option case ($c = 0$ & $\kappa = 0$) are consistently lower than cases where we have nonzero fees and surrender charges.

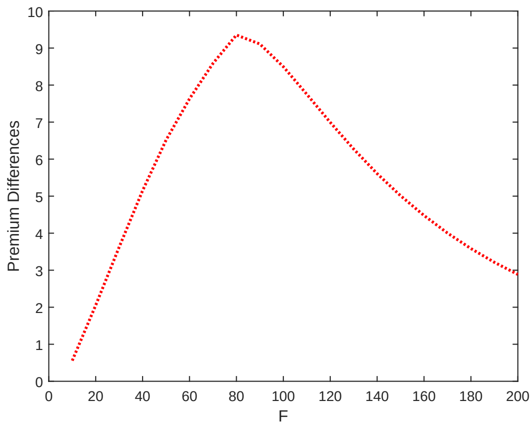


Figure 5: Premium Differences which is equal to the surrender option value minus the standard American call option value.

We sum up the section by presenting the sensitivities of guarantee premiums to changes on the underlying fund value for maturities ranging from 6 months to 15 years in Table 3. We note that deltas for deep in-the-money guarantees with shorter maturities are very close to -1 implying that for every \$1 increase in the fund value, the guarantee premium will decrease by \$1. For deep-in-the-money guarantees, the deltas gradually drift from -1 with increasing maturities. This behaviour is reversed for out-of-the-money guarantees whose deltas become more negative with increasing maturities.

Fund Value	$c = 0$, $\kappa = 0$	$c = 0.01$, $\kappa = 0.01$	$c = 0.03$, $\kappa = 0$	$c = 0.03$, $\kappa = 0.02$	$c = 0.03$, $\kappa = 0.03$
40	60	65.2470	60	70.2339	74.7314
50	50	56.6159	50	62.8030	68.4884
60	40	47.7468	40	54.8843	61.6104
70	30	38.5665	30.9631	46.3405	53.8201
80	21.3315	29.6139	24.5171	37.7474	45.2010
90	15.6349	22.5599	19.8098	30.6772	37.2821
100	11.7707	17.5265	16.2646	25.2593	31.0000
110	9.0604	13.8665	13.5269	21.0425	26.0470
120	7.1019	11.1369	11.3704	17.7024	22.0809
130	5.6521	9.0587	9.6440	15.0185	18.8631
140	4.5569	7.4486	8.2431	12.8354	16.2234
150	3.7152	6.1828	7.0934	11.04101	14.0371
160	3.0588	5.1749	6.1406	9.5526	12.2110

Table 2: Premium values when $G = 100$ with all other parameters as presented in Table 1.

Fund Value	$T = 0.5$	$T = 1$	$T = 5$	$T = 10$	$T = 15$
20	-0.99484791	-0.989725298	-0.949623193	-0.897362991	-0.845196661
40	-0.99484791	-0.989723709	-0.943734302	-0.892194197	-0.853653871
60	-0.994806799	-0.988795379	-0.952461492	-0.926987922	-0.903160608
80	-0.994181179	-0.945265697	-0.734921426	-0.691619866	-0.686952558
100	-0.456332196	-0.440967322	-0.397656989	-0.38784721	-0.39079854
120	-0.076696552	-0.135035882	-0.21952068	-0.233500335	-0.241847212
140	-0.005844514	-0.029891602	-0.122438974	-0.147318762	-0.158547139
160	-0.000263757	-0.005346292	-0.068960555	-0.096189125	-0.108387754

Table 3: Delta values when $G = 100$ with all other parameters as presented in Table 1.

6 Conclusion

In this paper we have presented a numerical integration technique for valuing surrender options in guaranteed minimum maturity benefits embedded in variable annuity contracts. We formulate the problem using optimal stopping theory and then present a systematic approach of transforming the optimal stopping time problem into a free-boundary problem. Jamshidian (1992) techniques are employed to transform the homogenous free-boundary problem to a non-homogeneous partial differential equation (PDE). An integral expression has been presented as the general solution by using Duhamel's principle and this is a function of the transition density function. The transition density function is a solution of the of the associated Kolmogorov backward PDE whose solution is well known in literature.

Semi-closed form expressions for the integral terms of the price and the corresponding delta of the surrender option have been derived and implemented using Simpson's rule. Numerical results exploring the impact of surrender fees and insurance charges on the surrender option prices, free-boundary and the delta have been provided. Efficient plots and tabular results have been presented to emphasize the impact surrender fees, insurance charges and other variable parameters on the early exercise boundaries, prices and the delta of the surrender options. Numerical comparisons with standard American put option prices have been presented and we have generally noted that surrender fees and the continuously compounded charges make the premiums for surrender options more valuable as compared to standard American put options.

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A Appendices

A.1 Proof of Proposition 3.2

We first derive the explicit form of the European option component, $V_E(\tau, x)$, as follows

$$
\begin{aligned}
V_E(\tau,x) &= e^{-r\tau}\int_{-\infty}^{\infty}(G-e^{w})^{+}U(\tau,x;w)dw \\
&= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\ln G}(G-e^{w})\exp\left\{-\frac{x-w+\phi\tau}{2\sigma^2\tau}\right\}dw \\
&\equiv A_1(\tau,x)-A_2(\tau,x), \qquad (45)
\end{aligned}
$$

where

$$
A_1(\tau,x)=\frac{e^{-r\tau}}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\ln G}G\exp\left\{-\frac{x-w+\phi\tau}{2\sigma^2\tau}\right\}dw, \qquad (46)
$$

and

$$
A_2(\tau,x)=\frac{e^{-r\tau}}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\ln G}e^{w}\exp\left\{-\frac{x-w+\phi\tau}{2\sigma^2\tau}\right\}dw. \qquad (47)
$$

In simplifying $A_1(\tau,x)$, we let $y=\frac{x-w+\phi\tau}{\sigma\sqrt{\tau}}$ such that $dw=-\sigma\sqrt{\tau}dy$. Also

$$
w=\ln G\Rightarrow y=\frac{x-\ln G+\phi\tau}{\sigma\sqrt{\tau}} \quad \text{and} \quad w=-\infty\Rightarrow y=\infty.
$$

Equation (46) then becomes

$$
A_1(\tau,x)=\frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{-d_2}^{\infty}Ge^{-\frac{y^2}{2}}dy=Ge^{-r\tau}\mathcal{N}\left(-d_2\left(\tau,x,G\right)\right), \qquad (48)
$$

where $\mathcal{N}\left(-d_2\left(\tau,x,G\right)\right)$ is a cumulative Normal distribution function with

$$
d_2(\tau,x,K)=\frac{x-\ln G+\phi\tau}{\sigma\sqrt{\tau}}. \qquad (49)
$$

The second component, $A_2(\tau,x)$ is simplified by first re-writing it as follows

$$
A_2(\tau,x)=\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{\ln G}\exp\left\{w-\frac{(x-w+\phi\tau)^2}{2\sigma^2\tau}\right\}dw.
$$

By completing the square and simplifying the above equation we obtain

$$
A_2(\tau,x)=\frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{\ln G}\exp\left\{\frac{[x+\phi\tau]^2}{-2\sigma^2\tau}\right\}\exp\left\{\frac{w^2-2w[x+\{r-c+\frac{1}{2}\sigma^2\}\tau]}{-2\sigma^2\tau}\right\}dw, \qquad (50)
$$

which can also be represented as

$$
\begin{aligned}
A_2(\tau,x) &= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{\ln G}\exp\left\{\frac{[x+\phi\tau]^2}{-2\sigma^2\tau}\right\}\exp\left\{\frac{[x+(r-c+\frac{1}{2}\sigma^2)\tau]^2}{2\sigma^2\tau}\right\}\exp\left\{\frac{[x-w+(r-c+\frac{1}{2}\sigma^2)\tau]^2}{-2\sigma^2\tau}\right\}dw \\
&= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{\ln G}e^{x+(r-c)\tau}\exp\left\{\frac{[x-w+(r-c+\frac{1}{2}\sigma^2)\tau]^2}{-2\sigma^2\tau}\right\}dw \\
&= \frac{e^{-c\tau}e^{x}}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{\ln G}\exp\left\{\frac{[x-w+(r-c+\frac{1}{2}\sigma^2)\tau]^2}{-2\sigma^2\tau}\right\}dw. \qquad (51)
\end{aligned}
$$

Now, we let

$$
y=\frac{x-w+(r-c+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\Rightarrow dw=-\sigma\sqrt{\tau}.
$$

As for the integral limits, $w = \ln G \Rightarrow y = \frac{x - \ln G + (r - c + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$ and $w = -\infty \Rightarrow y = \infty$, hence

$$A_2(\tau, x) = \frac{e^{-c\tau} e^x}{2\pi} \int_{-d_1}^{\infty} e^{-\frac{y^2}{2}} dy = e^{-c\tau} e^x \mathcal{N}\left(-d_1\left(\tau, x, G\right)\right), \tag{52}$$

with

$$d_1(\tau, x, G) = \frac{x - \ln G + (r - c + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}. \tag{53}$$

By comparing equations (53) and (49) it can be shown that

$$d_2(\tau, x, G) = d_1(\tau, x, G) - \sigma\sqrt{\tau}. \tag{54}$$

Combining the results in equations (48) and (52) yield the European option component presented in equation (29) of Proposition 3.2.

Next we derive the explicit form of the early exercise premium by simplifying the expression presented in equation (26) which we reproduce here as

$$V_P(\tau, x) = \int_0^{\tau} e^{-r(\tau-\xi)} \int_{-\infty}^{\ln B_\xi + \kappa(\tau-\xi)} [rG - (c-k)e^{-\kappa(\tau-\xi)} e^w] U(\tau - \xi, x; w) dw d\xi. \tag{55}$$

The derivations proceed as those for the European option component case. We split the above equation in two parts by letting

$$V_P(\tau, x) = I(\tau, x) - II(\tau, x), \tag{56}$$

where

$$I(\tau, x) = \int_0^{\tau} e^{-r(\tau-\xi)} \int_{-\infty}^{\ln B_\xi + \kappa(\tau-\xi)} rG \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}} \exp\left\{-\frac{(x - w + \phi(\tau-\xi))^2}{2\sigma^2(\tau-\xi)}\right\} dw d\xi, \tag{57}$$

and

$$II(\tau, x) = \int_0^{\tau} e^{-r(\tau-\xi)} \int_{-\infty}^{\ln B_\xi + \kappa(\tau-\xi)} (c-k)e^{-\kappa(\tau-\xi)} e^w \frac{1}{\sigma\sqrt{2\pi(\tau-\xi)}} \exp\left\{-\frac{(x - w + \phi(\tau-\xi))^2}{2\sigma^2(\tau-\xi)}\right\} dw d\xi. \tag{58}$$

In simplifying the first component, $I(\tau, x)$, we let $y = \frac{x - w + \phi(\tau-\xi)}{\sigma\sqrt{\tau-\xi}}$, such that $dw = -\sigma\sqrt{\tau-\xi}dy$. Also

$$w = \ln B + \kappa(\tau-\xi) \Rightarrow y = \frac{x - \ln B - \kappa(\tau-\xi) + \phi(\tau-\xi)}{\sigma\sqrt{\tau-\xi}}$$

and $w = -\infty \Rightarrow y = \infty$. Equation (57) then becomes

$$I(\tau, x) = \int_0^{\tau} \frac{e^{-r(\tau-\xi)}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} rGe^{-\frac{y^2}{2}} dy d\xi = rG \int_0^{\tau} e^{-r(\tau-\xi)} \mathcal{N}\left(-d_2\left(\tau-\xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right) d\xi. \tag{59}$$

The second component, $II(\tau, x)$, is simplified by first rearranging it as follows

$$II(\tau, x) = \int_0^{\tau} \frac{(c-\kappa)e^{-r(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\ln B + \kappa(\tau-\xi)} e^{-\kappa(\tau-\xi)} \exp\{w - \frac{(x - w + \phi(\tau-\xi))^2}{2\sigma^2(\tau-\xi)}\} dw d\xi.$$

By completing the square and simplifying the above equation we obtain

$$\begin{aligned} II(\tau, x) = \int_0^{\tau} \frac{(c-\kappa)e^{-(r+\kappa)(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}} &\int_{-\infty}^{\ln B + \kappa(\tau-\xi)} \exp\left\{\frac{[x + \phi(\tau-\xi)]^2}{-2\sigma^2(\tau-\xi)}\right\} \\ &\times \exp\left\{\frac{w^2 - 2w[x + \{r - c + \frac{1}{2}\sigma^2\}(\tau-\xi)]}{-2\sigma^2(\tau-\xi)}\right\} dw d\xi, \end{aligned} \tag{60}$$

which can also be represented as

$$
\begin{aligned}
II(\tau,x) = & \int_0^\tau \frac{(c-\kappa)e^{-(r+\kappa)(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\ln B+\kappa(\tau-\xi)} \exp\left\{\frac{[x+\phi(\tau-\xi)]^2}{-2\sigma^2(\tau-\xi)}\right\} \exp\left\{\frac{[x+(r-c+\frac{1}{2}\sigma^2)(\tau-\xi)]^2}{2\sigma^2(\tau-\xi)}\right\} \\
& \times \exp\left\{\frac{[x-w+(r-c+\frac{1}{2}\sigma^2)(\tau-\xi)]^2}{-2\sigma^2(\tau-\xi)}\right\} dwd\xi \\
= & \int_0^\tau \frac{(c-\kappa)e^{-(r+\kappa)(\tau-\xi)}}{\sigma\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\ln B+\kappa(\tau-\xi)} \exp\{x+(r-c)(\tau-\xi)\} \\
& \times \exp\left\{\frac{[x-w+(r-c+\frac{1}{2}\sigma^2)(\tau-\xi)]^2}{-2\sigma^2(\tau-\xi)}\right\} dwd\xi \qquad (61) \\
= & \int_0^\tau \frac{(c-\kappa)e^x}{\sigma\sqrt{2\pi(\tau-\xi)}} \int_{-\infty}^{\ln B+\kappa(\tau-\xi)} \exp\{-(c+\kappa)(\tau-\xi)\} \exp\left\{\frac{[x-w+(r-c+\frac{1}{2}\sigma^2)(\tau-\xi)]^2}{-2\sigma^2(\tau-\xi)}\right\} dwd\xi.
\end{aligned}
$$

Now, we let

$$
y = \frac{x-w+(r-c+\frac{1}{2}\sigma^2)(\tau-\xi)}{\sigma\sqrt{(\tau-\xi)}} \Rightarrow dw = -\sigma\sqrt{(\tau-\xi)}.
$$

When $w = \ln B+\kappa(\tau-\xi) \Rightarrow y = \frac{x-\ln B-\kappa(\tau-\xi)+(r-c+\frac{1}{2}\sigma^2)(\tau-\xi)}{\sigma\sqrt{(\tau-\xi)}}$ and $w=-\infty \Rightarrow y=\infty$, hence

$$
\begin{aligned}
II(\tau,x) &= \int_0^\tau \frac{(c-\kappa)e^x}{2\pi} \int_{-d_1}^{\infty} e^{(c+\kappa)(\tau-\xi)} e^{-\frac{y^2}{2}} dyd\xi \\
&= (c-\kappa)e^x \int_0^\tau e^{-(c+\kappa)(\tau-\xi)} \mathcal{N}\left(-d_1\left(\tau-\xi, x, B_\xi e^{\kappa(\tau-\xi)}\right)\right) d\xi. \qquad (62)
\end{aligned}
$$

Combining equations (59) and (62) yields the results presented in (30).

A.2 Proof of Proposition 3.3

Rearrange equation (32) as

$$
\begin{aligned}
\frac{B_\tau}{G} = & \left[e^{-r\tau}\mathcal{N}(-d_2(\tau,\ln B_\tau+\kappa\tau,G)) + r\int_0^\tau e^{-r(\tau-\xi)}\mathcal{N}\left(-d_2\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\right)d\xi - 1\right] \\
& \times \Bigg[e^{-(c-\kappa)\tau}\mathcal{N}(-d_1(\tau,\ln B_\tau+\kappa\tau,G)) + (c-\kappa)e^{\kappa\tau} \\
& \quad \times \int_0^\tau e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}\left(-d_1\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\right)d\xi - 1\Bigg]^{-1} \qquad (63)
\end{aligned}
$$

For simplicity, we let

$$
M_1(\tau) = e^{-r\tau}\mathcal{N}(-d_2(\tau,\ln B_\tau+\kappa\tau,G)) + r\int_0^\tau e^{-r(\tau-\xi)}\mathcal{N}\left(-d_2\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\right)d\xi - 1, \qquad (64)
$$

and

$$
\begin{aligned}
M_2(\tau) = & e^{-(c-\kappa)\tau}\mathcal{N}(-d_1(\tau,\ln B_\tau+\kappa\tau,G)) \qquad (65) \\
& + (c-\kappa)e^{\kappa\tau}\int_0^\tau e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}\left(-d_1\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\right)d\xi - 1.
\end{aligned}
$$

Next we wish to find $\lim_{\tau\to 0}\frac{B_\tau}{G}$ with the aid of l'Hôpital's rule. To this end, we first calculate the derivatives of $M_1(\tau)$ and $M_2(\tau)$ as

$$\begin{aligned}M_1'(\tau)=&-re^{-r\tau}\mathcal{N}(-d_2(\tau,\ln B_\tau+\kappa\tau,G))-e^{-r\tau}\mathcal{N}'(-d_2(\tau,\ln B_\tau+\kappa\tau,G))\frac{\partial}{\partial\tau}d_2(\tau,\ln B_\tau+\kappa\tau,G)\\&+r\mathcal{N}\left(-d_2\left(0,\ln B_\tau+\kappa\tau,B_\tau\right)\right)+r\int_0^\tau\Big\{-re^{-r(\tau-\xi)}\mathcal{N}\left(-d_2\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\right)\\&-e^{-r(\tau-\xi)}\mathcal{N}'\left(-d_2\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{\partial}{\partial\tau}d_2\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\Big\}d\xi,\end{aligned}\tag{66}$$

and

$$\begin{aligned}M_2'(\tau)=&-(c-\kappa)e^{-(c-\kappa)\tau}\mathcal{N}(-d_1(\tau,\ln B_\tau+\kappa\tau,G))-e^{-(c-\kappa)\tau}\mathcal{N}'(-d_1(\tau,\ln B_\tau+\kappa\tau,G))\frac{\partial}{\partial\tau}d_1(\tau,\ln B_\tau+\kappa\tau,G)\\&+(c-\kappa)e^{\kappa\tau}\mathcal{N}\left(-d_1\left(0,\ln B_\tau+\kappa\tau,B_\tau\right)\right)\qquad(67)\\&+(c-\kappa)e^{\kappa\tau}\int_0^\tau\Big\{-ce^{-(c+\kappa)(\tau-\xi)}\mathcal{N}\left(-d_1\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\right)\\&-e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}'\left(-d_1\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{\partial}{\partial\tau}d_1\left(\tau-\xi,\ln B_\tau+\kappa\tau,B_\xi e^{\kappa(\tau-\xi)}\right)\Big\}d\xi.\end{aligned}$$

For $i=1,2$, we notice that

$$\lim_{\tau\to 0}d_i(\tau,\ln B_\tau+\kappa\tau,G)=\begin{cases}0, & \text{if } B_0=G,\\ \infty, & \text{if } B_0>G.\end{cases}\tag{68}$$

So if $B_0>G$, we have

$$\lim_{\tau\to 0}M_1'(\tau)=\frac{r}{2},\tag{69}$$

and

$$\lim_{\tau\to 0}M_2'(\tau)=\frac{c-\kappa}{2}.\tag{70}$$

Therefore, taking limit in (63) and using l'Hôpital's rule yield

$$B_0=\min\left(1,\frac{r}{c-\kappa}\right)G.\tag{71}$$

A.3 Proof of Proposition 3.4

The derivation for $D_E(\tau,x)$ is the same as that for delta of a European put option. We only derive $D_P(\tau,x)$. Differentiating $V_P(\tau,x)$ with respect to the underlying fund value yields

$$\begin{aligned}D_P(\tau,x)=&-rG\int_0^\tau e^{-r(\tau-\xi)}\mathcal{N}'\left(-d_2\left(\tau-\xi,x,B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{\partial}{\partial F}d_2\left(\tau-\xi,x,B_\xi e^{\kappa(\tau-\xi)}\right)d\xi\\&-(c-\kappa)\int_0^\tau e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}\left(-d_1\left(\tau-\xi,x,B_\xi e^{\kappa(\tau-\xi)}\right)\right)d\xi\\&+(c-\kappa)e^x\int_0^\tau e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}'\left(-d_1\left(\tau-\xi,x,B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{\partial}{\partial F}d_1\left(\tau-\xi,x,B_\xi e^{\kappa(\tau-\xi)}\right)d\xi\\=&-\frac{rG}{\sigma F}\int_0^\tau e^{-r(\tau-\xi)}n\left(-d_2\left(\tau-\xi,x,B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{1}{\sqrt{\tau-\xi}}d\xi\\&-(c-\kappa)\int_0^\tau e^{-(c+\kappa)(\tau-\xi)}\mathcal{N}\left(-d_1\left(\tau-\xi,x,B_\xi e^{\kappa(\tau-\xi)}\right)\right)d\xi\\&+\frac{c-\kappa}{\sigma}\int_0^\tau e^{-(c+\kappa)(\tau-\xi)}n\left(-d_1\left(\tau-\xi,x,B_\xi e^{\kappa(\tau-\xi)}\right)\right)\frac{1}{\sqrt{\tau-\xi}}d\xi.\end{aligned}\tag{72}$$