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Managing Systematic Mortality Risk in Life Annuities: An Application of Longevity Derivatives

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Abstract

Developing a liquid longevity market requires reliable and well-designed financial instruments. An index-based longevity swap and a cap are analysed in this paper under a tractable stochastic mortality model. The model is calibrated using Australian mortality data and analytical formulas for prices of longevity derivatives are provided. Hedge effectiveness is examined under a hypothetical life annuity portfolio subject to longevity risk. The paper presents various hedging features exhibited by a longevity swap and a cap based on different assumptions underlying the market price of longevity risk, the term to maturity of hedging instruments, as well as the size of the underlying annuity portfolio. The results are demonstrated to have important implications for the optimal use of longevity hedging instruments with linear and nonlinear payoff structures.

Key words: longevity risk management; longevity swaps; longevity options; hedge effectiveness

1 Introduction

Securing a comfortable living after retirement is perhaps one of the main goals for the majority of the working population around the world. A major risk in retirement, however, is the possibility that retirement savings will be outlived. Products that provide guaranteed lifetime income, such as life annuities, are designed to shift the so-called longevity risk, that is, the risk of living longer than expected, to the product providers. Annuity providers and pension funds are the main sellers of such policies. A major concern for many of these market players is how to manage systematic mortality risk [1], associated with random changes in the underlying mortality intensity, in a life annuity portfolio. Systematic mortality risk cannot be diversified away with increasing portfolio size, while idiosyncratic mortality risk, representing the randomness of deaths in a portfolio with fixed mortality intensity, is diversifiable.

A possible way to mitigate longevity risk is through reinsurance. However, it turns out that reinsurers have a small risk appetite and are reluctant to take this "toxic" risk (Blake et al. (2006b)). In fact, even if they were willing to accept the risk, the reinsurance sector is not deep enough to absorb the vast scale of longevity risk currently undertaken by annuity providers and pension funds. [2] Due to the sheer size of capital markets and almost zero correlation between financial and demographic risks, managing longevity risk through capital markets is arguably the best solution available. The first generation of capital market solutions for longevity risk, provided by mortality and longevity bonds (Blake and Burrows (2001), Blake et al. (2006a) and Bauer et al. (2010)) [3], was proposed with variable success. The second generation involving forwards and swaps proved to be a more welcomed solution for longevity risk management among practitioners (Blake et al. (2013)). Examples of longevity derivatives with a linear payoff include q-forwards and S-forwards, which have as underlying the mortality and the survival rate, respectively (LLMA (2010a)). Studies on longevity derivatives with a nonlinear payoff structure have only been partially covered in the literature. Boyer and Stentoft (2013) evaluate European and American type survivor options via simulation. Wang and Yang (2013) propose and price survivor floors under an extension of the Lee-Carter model. However, they do not consider hedge effectiveness of longevity options used as hedging instruments.

This paper provides pricing analysis of longevity derivatives, as well as their hedge effectiveness and hedging features using a hypothetical life annuity portfolio subject to longevity risk, under a continuous time modelling framework. A longevity swap and a cap are chosen as representatives of linear and nonlinear products respectively to illustrate a market-based longevity risk management. Ngai and Sherris (2011) study the effectiveness of static hedging of longevity risk in different annuity portfolios using a range of longevity-linked instruments. Specifically, q-forwards, longevity bonds and longevity swaps are used as hedging instruments to mitigate longevity risk. Hari et al. (2008) apply a generalised two-factor Lee-Carter model to investigate the impact of longevity risk on the solvency of pension annuities. Li and Hardy (2011) consider hedging longevity risk via a portfolio of q-forwards and measure basis risk, which is one of the major obstacles in the development of a longevity market. A major goal of the current paper is to demonstrate various hedging features exhibited by a longevity swap and a cap based on different underlying assumptions for the market price of longevity risk, the term to maturity of hedging instruments, as well as the size of the underlying annuity portfolio.

[1] From an annuity provider's perspective, longevity risk modelling can lead to an (stochastically) over- or underestimation of survival probabilities for all annuitants. For this reason longevity risk is also referred to as the systematic mortality risk.

[2] It is estimated that pension assets for the 13 largest major pension markets have reached nearly 30 trillions in 2012 (Global Pension Assets Study 2013, Towers Watson).

[3] Of particular interest is an attempt to issue the EIB longevity bond by the European Investment Bank (EIB) in 2004, which was underwritten by BNP Paribas. This bond was not well received by investors and could not generate enough demand to be launched due to its deficiencies, as outlined in Blake et al. (2006a).

As dynamic hedging requires a liquid longevity market which is yet to be developed, we consider static hedging using an index-based longevity swap and a cap. Index-based instruments are designed specifically to mitigate systematic mortality risk, and are supposed to be less costly and convenient for trading as they are standardised contracts (Blake et al. (2013)). However, unlike bespoke or customized hedging instruments, they do not cover idiosyncratic mortality risk and may give rise to population basis risk (Li and Hardy (2011)). Portfolio size is thus one of the main factors that determines the hedge effectiveness of index-based instruments. Cairns (2011) applies q-forwards as hedging instruments to construct a discrete-time delta hedging strategy, and compares it with static hedging using different numbers of q-forwards maturing in different years. In this paper the effect of the term to maturity of a longevity swap and a cap, which are constructed as a portfolio of S-forwards and longevity caplets, respectively, is analysed while market price of longevity risk is taken into account. The results are shown to have important implications for the optimal use of longevity hedging instruments with linear and nonlinear payoff structures under different situations.

As pointed out in Cairns (2011), the lack of analytical formulas for pricing q-forwards and its derivatives, known as "Greeks", has posed a significant problem as simulations within simulations are required when examining dynamic hedging strategies and their effectiveness. In this paper we utilise a tractable stochastic mortality model under which survival probabilities and prices of longevity derivatives can be derived analytically. The importance of tractable models has been emphasised in Luciano et al. (2012) who study how dynamic hedging can effectively be used for longevity and interest rate risk.

The paper is organised as follows. Section 2 specifies a tractable two-factor Gaussian mortality model, and its parameters are estimated using Australian males mortality data. Section 3 analyses longevity derivatives, in particular, a longevity swap and a cap, from a pricing perspective. Explicit pricing formulas are derived under the proposed two-factor Gaussian mortality model. Section 4 examines various hedging features and hedge effectiveness of a longevity swap and a cap on a hypothetical life annuity portfolio exposed to longevity risk. Section 5 summarises the results and provides concluding remarks.

2 Mortality Model

Let $(\Omega, \mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t, \mathbb{P})$ be a filtered probability space where $\mathbb{P}$ is the real world probability measure. The subfiltration $\mathcal{G}_t$ contains information about the dynamics of the mortality intensity while death times of individuals are captured by $\mathcal{H}_t$. It is assumed that the interest rate r is constant where $B(0,t) = e^{-r\,t}$ denotes the price of a t-year zero coupon bond, and our focus is on the modelling of stochastic mortality.

2.1 Model Specification

For the purpose of financial risk management applications one requires stochastic mortality model that is tractable, and is able to capture well the mortality dynamics for different ages. We work under the affine mortality intensity framework and assume the mortality intensity to be Gaussian such that analytical prices can be derived for longevity options, as described in Section 3. Gaussian mortality models have been considered in Bauer et al. (2010) and Blackburn and Sherris (2013) within the forward mortality framework. Luciano and Vigna (2008) suggest Gaussian mortality where the intensity follows the Ornstein-Uhlenbeck process. In addition, Jevtic et al. (2013) consider a continuous time cohort model where the underlying mortality dynamics is Gaussian.

We consider a two-factor Gaussian mortality model for the mortality intensity process $\mu_{x+t}(t)$ of

a cohort aged x at time $t = 0$ [4]:

$$d\mu_x(t) = dY_1(t) + dY_2(t), \tag{2.1}$$

where

$$dY_1(t) = \alpha_1 Y_1(t)\, dt + \sigma_1\, dW_1(t) \tag{2.2}$$
$$dY_2(t) = (\alpha\, x + \beta)\, Y_2(t)\, dt + \sigma e^{\gamma x}\, dW_2(t) \tag{2.3}$$

and $dW_1 dW_2 = \rho\, dt$. The first factor $Y_1(t)$ is a general trend for the intensity process that is common to all ages. The second factor $Y_2(t)$ depends on the initial age through the drift and the volatility terms. [5] The initial values $Y_1(0)$ and $Y_2(0)$ of the factors are denoted by y_1 and y_2^x, respectively. The model is tractable and for a specific choice of the parameters (when $\alpha = \gamma = 0$) has been applied to short rate modelling in Brigo and Mercurio (2007).

Proposition 2.1 *Under the two-factor Gaussian mortality model (Eq.* (2.1) - (2.3)*), the* $(T-t)$*-year expected survival probability of a person aged* $x+t$ *at time* t*, conditional on filtration* $\mathcal{F}_t$*, is given by*

$$S_{x+t}(t,T) \stackrel{def}{=} E_t^{\mathbb{P}}\left(e^{-\int_t^T \mu_x(v)dv}\right) = e^{\frac{1}{2}\Gamma(t,T)-\Theta(t,T)}, \tag{2.4}$$

where, using $\alpha_2 = \alpha x + \beta$ *and* $\sigma_2 = \sigma e^{\gamma x}$*,*

$$\Theta(t,T) = \frac{(e^{\alpha_1(T-t)}-1)}{\alpha_1} Y_1(t) + \frac{(e^{\alpha_2(T-t)}-1)}{\alpha_2} Y_2(t) \ \ \textit{and} \tag{2.5}$$

$$\Gamma(t,T) = \sum_{k=1}^{2} \frac{\sigma_k^2}{\alpha_k^2}\left(T - t - \frac{2}{\alpha_k} e^{\alpha_k(T-t)} + \frac{1}{2\alpha_k} e^{2\alpha_k(T-t)} + \frac{3}{2\alpha_k}\right) + \frac{2\rho\sigma_1\sigma_2}{\alpha_1\alpha_2}\left(T - t - \frac{e^{\alpha_1(T-t)}-1}{\alpha_1} - \frac{e^{\alpha_2(T-t)}-1}{\alpha_2} + \frac{e^{(\alpha_1+\alpha_2)(T-t)}-1}{\alpha_1+\alpha_2}\right). \tag{2.6}$$

are the mean and the variance of the integral $\int_t^T \mu_x(v)\, dv$*, which is Gaussian distributed, respectively.*

We will use the fact that the integral $\int_t^T \mu_x(v)\, dv$ is Gaussian with known mean and variance to derive analytical pricing formulas for longevity options in Section 3.

Proof. Solving Eq. (2.2) to obtain an integral form of $Y_1(t)$, we have

$$\int_t^T Y_1(u)\, du = \int_t^T Y_1(t) e^{\alpha_1(u-t)} du + \int_t^T \sigma_1 \int_t^u e^{\alpha_1(u-v)} dW_1(v) du. \tag{2.7}$$

The first term in Eq. (2.7) can be simplified to

$$\int_t^T Y_1(t) e^{\alpha_1(u-t)} du = \frac{\left(e^{\alpha_1(T-t)}-1\right)}{\alpha_1} Y_1(t).$$

[4] For simplicity of notation we replace $\mu_{x+t}(t)$ by $\mu_x(t)$.

[5] We can in fact replace x by $x+t$ in Eq. (2.3). Using $x+t$ will take into account the empirical observation that the volatility of mortality tends to increase along with age $x+t$ (Figures 1 and 2). However, for a Gaussian process the intensity will have a non-negligible probability of reaching negative value when the volatility from the second factor ($\sigma e^{\gamma(x+t)}$) becomes very high, which occurs for example when $x+t > 100$ (given $\gamma > 0$). Using x instead of $x+t$ will also make the result in Section 3 easy to interpret. For these reasons we assume that the second factor $Y_2(t)$ depends on the initial age x only.

For the second term, we have

$$\sigma_1 \int_t^T e^{\alpha_1 u} \int_t^u e^{-\alpha_1 v} dW_1(v) du = \sigma_1 \int_t^T \int_t^u e^{-\alpha_1 v} dW_1(v) d_u \left(\frac{1}{\alpha_1} e^{\alpha_1 u} \right)$$
$$= \frac{\sigma_1}{\alpha_1} \int_t^T d_u \left(e^{\alpha_1 u} \int_t^u e^{-\alpha_1 v} dW_1(v) \right) - \frac{\sigma_1}{\alpha_1} \int_t^T e^{\alpha_1 u} d_u \left(\int_t^u e^{-\alpha_1 v} dW_1(v) \right)$$
$$= \frac{\sigma_1}{\alpha_1} e^{\alpha_1 T} \int_t^T e^{-\alpha_1 u} dW_1(u) - \frac{\sigma_1}{\alpha_1} \int_t^T e^{\alpha_1 u} e^{-\alpha_1 u} dW_1(u) = \frac{\sigma_1}{\alpha_1} \int_t^T e^{\alpha_1 (T-u)} - 1 \, dW_1(u),$$

where stochastic integration by parts is applied in the second equality.

To obtain an integral representation for $Y_2(t)$, we follow the same steps as above, replacing $Y_1(t)$ by $Y_2(t)$ in Eq. (2.7). It is then straightforward to notice that

$$\int_t^T \mu_x(u) \, du = \int_t^T Y_1(u) + Y_2(u) \, du \tag{2.8}$$

is a Gaussian random variable with mean $\Theta(t,T)$ (Eq. (2.5)) and variance $\Gamma(t,T)$ (Eq. (2.6)). Equation (2.4) is obtained by applying the moment generating function of a Gaussian random variable. □

2.2 Parameter Estimation

The discretised process, where the intensity is assumed to be constant over each integer age and calendar year, is approximated by the central death rates $m(x,t)$ (Wills and Sherris (2011)). Figure 1 displays Australian male central death rates $m(x,t)$ for years $t = 1970, 1971, \ldots, 2008$ and ages $x = 60, 61, \ldots, 95$. Figure 2 shows the difference of the central death rates $\Delta m(x,t) = m(x+1, t+1) - m(x,t)$. The variability of $\Delta m(x,t)$ is evidently increasing with increasing age x, which leads to the anticipation that $\gamma > 0$. Furthermore, for a fixed age x, there is a slight improvement in central death rates for more recent years, compared to the past.

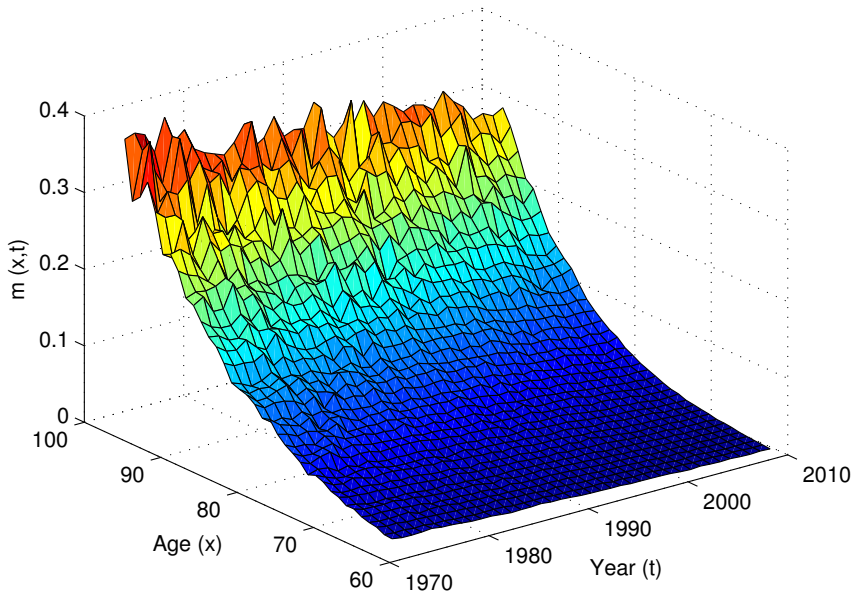

Fig. 1. Australian male central death rates $m(x,t)$ where $t = 1970, 1971, \ldots, 2008$ and $x = 60, 61, \ldots, 95$.

The parameters $\{\sigma_1, \sigma, \gamma, \rho\}$, which determine the volatility of the intensity process, are estimated as described below. As in Jevtic et al. (2013), we aim to estimate parameters using the method of

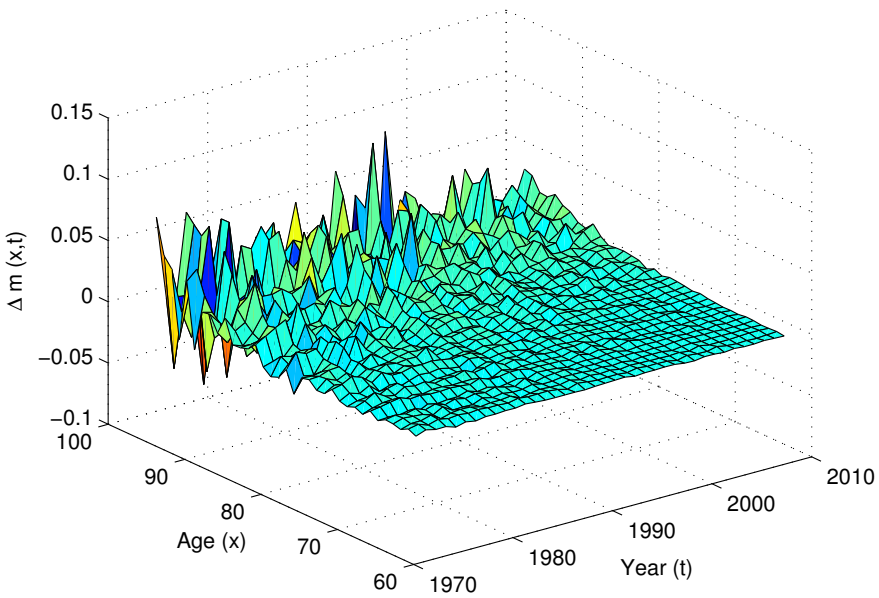

Fig. 2. Difference of the central death rates $\Delta m(x,t) = m(x+1,t+1) - m(x,t)$ where $t = 1970, 1971, \cdots, 2007$ and $x = 60, 61, \cdots, 94$.

least squares, thus, calibrating the model to the mortality surface. However, we take advantage of the fact that a Gaussian model is employed where the variance of the model can be calculated explicitly and thus, we capture the diffusion part of the process by matching the variance of the model to mortality data. Specifically, the implemented procedure is as specified below:

(1) Using empirical data for ages $x = 60, 65, \ldots, 90$ we evaluate the sample variance of $\Delta m(x,t)$ across time, denoted by $\mathrm{Var}(\Delta m_x)$.
(2) The model variance $\mathrm{Var}(\Delta\mu_x)$ for age x is given by

$$\begin{aligned}\mathrm{Var}(\Delta\mu_x) &= \mathrm{Var}(\sigma_1 \Delta W_1 + \sigma e^{\gamma x}\Delta W_2)\\ &= \left(\sigma_1^2 + 2\sigma_1\sigma\rho e^{\gamma x} + \sigma^2 e^{2\gamma x}\right)\Delta t.\end{aligned} \tag{2.9}$$

Since the difference between the death rates is computed in yearly terms, we set $\Delta t = 1$.
(3) The parameters $\{\sigma_1, \sigma, \gamma, \rho\}$ are then estimated by fitting the model variance $\mathrm{Var}(\Delta\mu_x)$ to the sample variance $\mathrm{Var}(\Delta m_x)$ for ages $x = 60, 65, \ldots, 90$ using least squares estimation, that is, by minimising

$$\sum_{x=60,65\ldots}^{90} \left(\mathrm{Var}(\Delta\mu_x|\sigma_1,\sigma,\gamma,\rho) - \mathrm{Var}(\Delta m_x)\right)^2 \tag{2.10}$$

with respect to the parameters $\{\sigma_1, \sigma, \gamma, \rho\}$.

The remaining parameters $\{\alpha_1, \alpha, \beta, y_1, y_2^{65}, y_2^{75}\}$ are then estimated as described below [6]:

(1) From the central death rates, we obtain empirical survival curves for cohorts aged 65 and 75 in 2008. The survival curve is obtained by setting

$$\hat{S}_x(0,T) = \prod_{v=1}^{T}(1 - m(x+v-1,0)) \tag{2.11}$$

[6] We calibrate the model for ages 65 and 75 simultaneously to obtain reasonable values for α and β since the drift of the second factor $Y_2(t)$ is age-dependent.

where $m(x,t)$ is the central death rate of an x years old at time t.[7]

(2) The parameters $\{\alpha_1, \alpha, \beta, y_1, y_2^{65}, y_2^{75}\}$ are then estimated by fitting the survival curves $(S_x(0,T))$ of the model to the empirical survival curves using least squares estimation, that is, by minimising

$$\sum_{x=65,75} \sum_{j=1}^{T_x} \left(\hat{S}_x(0,j) - S_x(0,j)\right)^2 \tag{2.12}$$

where $T_{65} = 31$ and $T_{75} = 21$, with respect to the parameters $\{\alpha_1, \alpha, \beta, y_1, y_2^{65}, y_2^{75}\}$.

The estimated parameters are reported in Table 1. Since $\gamma > 0$ we observe that the volatility of the process is higher for older (initial) age x.

Table 1
Estimated model parameters.

σ_1	σ	γ	ρ	α_1
0.0022465	0.0000002	0.129832	-0.795875	0.0017508
α	β	y_1	y_2^{65}	y_2^{75}
0.0000615	0.120931	0.0021277	0.0084923	0.0294695

The upper panel of Figure 3 shows the percentiles of the simulated mortality intensity for ages 65 and 75 in the left and the right panel, respectively. One observes that the volatility of the mortality intensity is higher for a 75 year old compared to a 65 year old. Corresponding survival probabilities are displayed in the lower panel of Figure 3, together with the 99% confidence bands computed pointwise. As it is pronounced from the figures, the two-factor Gaussian model specified above, despite its simplicity, produces reasonable mortality dynamics for ages 65 and 75.

[7] Here $t = 0$ represents calendar year 2008 and we approximate the 1-year survival probability $e^{-m(x+v-1,0)}$ by $1 - m(x+v-1,0)$.

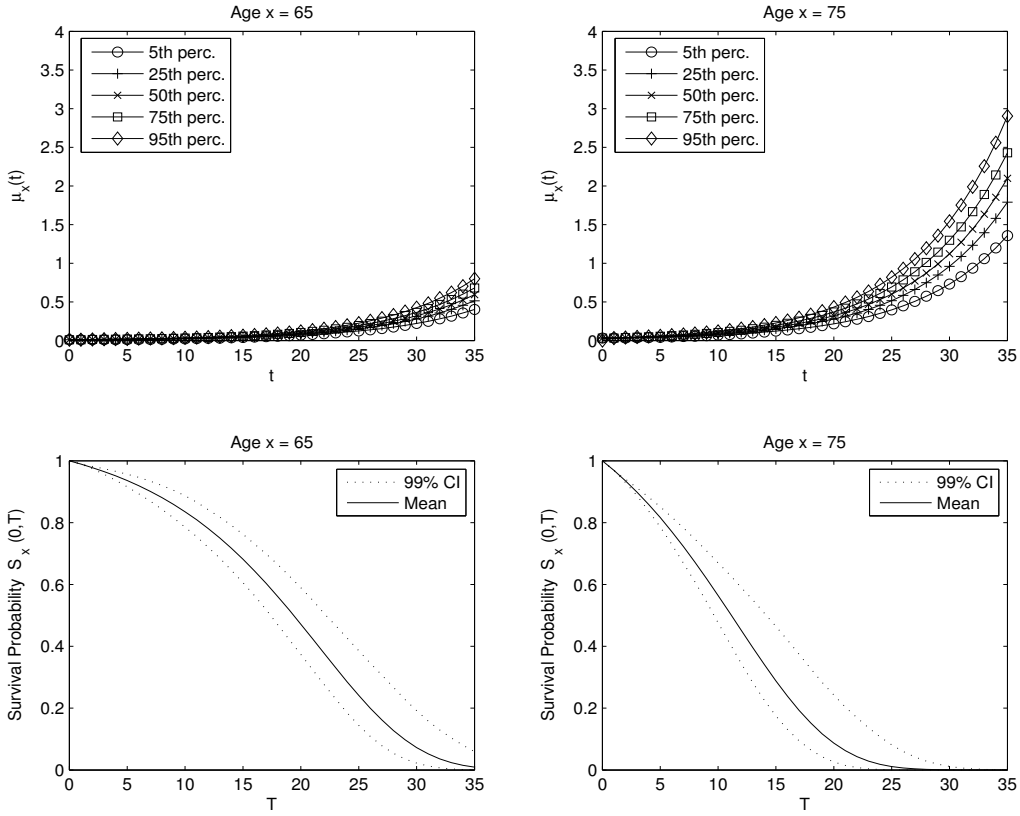

Fig. 3. Percentiles of the simulated intensity processes $\mu_{65}(t)$ and $\mu_{75}(t)$ for Australian males aged 65 (upper left panel) and 75 (upper right panel) in 2008, with their corresponding survival probabilities (the mean and the 99% confidence bands) for a 65 years old (lower left panel) and 75 years old (lower right panel).

3 Analytical Pricing of Longevity Derivatives

We consider longevity derivatives with different payoff structures including longevity swaps, longevity caps and longevity floors. Closed form expressions for prices of these longevity derivatives are derived under the assumption of the two-factor Gaussian mortality model introduced in Section 2. These instruments are written on survival probabilities and their properties are analysed from a pricing perspective.

3.1 Risk-Adjusted Measure

For the purpose of no-arbitrage valuation, we require the dynamics of the factors $Y_1(t)$ and $Y_2(t)$ to be written under a risk-adjusted measure.[8] To preserve the tractability of the model, we assume that the processes $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$ with dynamics

$$d\tilde{W}_1(t) = dW_1(t) \tag{3.1}$$

$$d\tilde{W}_2(t) = \lambda\sigma e^{\gamma x}\, Y_2(t)\, dt + dW_2(t) \tag{3.2}$$

[8] Since the longevity market is still in its development stage and hence, incomplete, we assume a risk-adjusted measure exists but is not unique.

are standard Brownian motions under a risk-adjusted measure $\mathbb{Q}$. In Eq. (3.2) λ represents the market price of longevity risk.[9] Under $\mathbb{Q}$ we can write the factor dynamics as follows:

$$dY_1(t) = \alpha_1 Y_1(t)\,dt + \sigma_1\,d\tilde{W}_1(t) \tag{3.3}$$

$$dY_2(t) = (\alpha x + \beta - \lambda\sigma e^{\gamma x})\,Y_2(t)\,dt + \sigma_2\,d\tilde{W}_2(t). \tag{3.4}$$

The corresponding risk-adjusted survival probability is given by

$$\tilde{S}_{x+t}(t,T) \stackrel{\text{def}}{=} E_t^{\mathbb{Q}}\left(e^{-\int_t^T \mu_x(v)\,dv}\right) = e^{\frac{1}{2}\tilde{\Gamma}(t,T)-\tilde{\Theta}(t,T)} \tag{3.5}$$

where $\alpha_2 = \alpha x + \beta$ is replaced by $(\alpha x + \beta - \lambda\sigma e^{\gamma x})$ in the expressions for $\tilde{\Theta}(t,T)$ and $\tilde{\Gamma}(t,T)$, see Eq. (2.5) and Eq. (2.6), respectively.

Since a liquid longevity market is yet to be developed, we aim to determine a reasonable value for λ based on the longevity bond announced by BNP Paribas and European Investment Bank (EIB) in 2004 as proposed in Cairns et al. (2006) and applied in Meyricke and Sherris (2014), see also Wills and Sherris (2011). The BNP/EIB longevity bond is a 25-year bond with coupon payments linked to a survivor index based on the realised mortality rates.[10] The price of the longevity bond is given by

$$V(0) = \sum_{T=1}^{25} B(0,T)\,e^{\delta T} E_0^{\mathbb{P}}\left(e^{-\int_0^T \mu_x(v)\,dv}\right) \tag{3.6}$$

where δ is a spread, or an average risk premium per annum[11], and the T-year projected survival rate is assumed to be the T-year survival probability for the Australian males cohort aged 65 as modelled in Section 2, see Eq. (2.4). Since the BNP/EIB bond is priced based on a yield of 20 basis points below standard EIB rates (Cairns et al. (2006)), we have the spread of $\delta = 0.002$.[12]

Under a risk-adjusted measure $\mathbb{Q}(\lambda)$, the price of the longevity bond corresponds to

$$V^{\mathbb{Q}(\lambda)}(0) = \sum_{T=1}^{25} B(0,T)\,E_0^{\mathbb{Q}(\lambda)}\left(e^{-\int_0^T \mu_x(v)\,dv}\right). \tag{3.7}$$

Fixing the interest rate to $r = 4\%$, we find a model-dependent λ, such that the risk-adjusted bond price $V^{\mathbb{Q}(\lambda)}(0)$ matches the market bond price $V(0)$ as close as possible. For example, for $\lambda = 8.5$ we have $V(0) = 11.9045$ and $V^{\mathbb{Q}(\lambda)}(0) = 11.9068$. For more details on the above procedure refer to Meyricke and Sherris (2014). In the following we assume that the risk-adjusted measure $\mathbb{Q}$ is determined by a unique value of λ.

Figure 4 shows the risk-adjusted survival probabilities for Australian males aged 65 with respect to different values of the market price of longevity risk λ. As one observes from the figure, a larger (positive) value of λ leads to an improvement in survival probability, while a smaller values of λ indicate a decline in survival probability under the risk-adjusted measure $\mathbb{Q}$.

[9] For simplicity, we assume that there is no risk adjustment for the first factor Y_1 and λ is age-independent.

[10] The issue price was determined by BNP Paribas using anticipated cash flows based on the 2002-based mortality projections provided by the UK Government Actuary's Department.

[11] The spread δ depends on the term of the bond and the initial age of the cohort being tracked (Cairns et al. (2006)), and δ is related to but distinct from λ, the market price of longevity risk.

[12] The reference cohort for the BNP/EIB longevity bond is the England and Wales males aged 65 in 2003. Since the longevity derivatives market is under-developed in Australia, we assume that the same spread of $\delta = 0.002$ (as in the UK) is applicable to the Australian males cohort aged 65 in 2008. Note however that sensitivity analyses will be performed in Section 4.

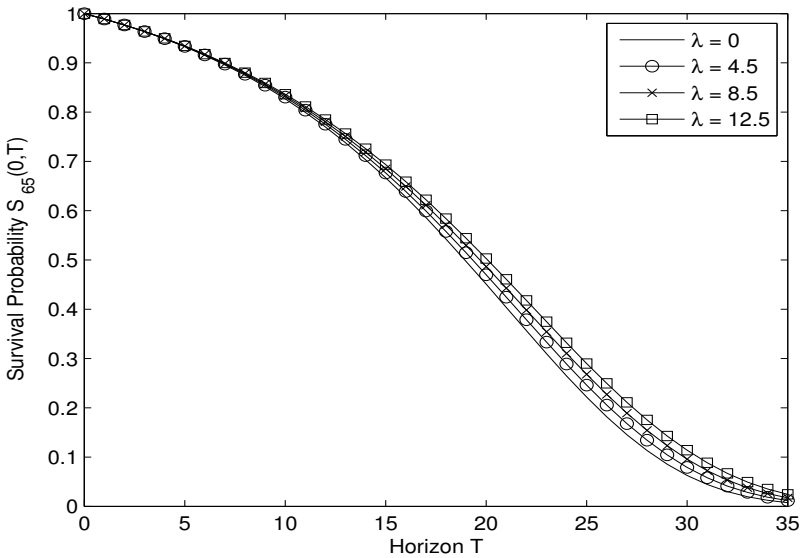

Fig. 4. Risk-adjusted survival probability with respect to different market price of longevity risk λ.

3.2 Longevity Swaps

A longevity swap involves counterparties swapping fixed payments for payments linked to the number of survivors in a reference population in a given time period, and can be thought of as a portfolio of S-forwards, see Dowd (2003). An S-forward, or 'survivor' forward has been developed by LLMA (2010b). Longevity swaps can be regarded as a stream of S-forwards with different maturity dates. One of the advantages of using S-forwards is that there is no initial capital requirement at the inception of the contract and cash flows occur only at maturity.

Consider an annuity provider who has an obligation to pay an amount dependent on the number of survivors, and hence, survival probability of a cohort at time T. If longevity risk is present, the survival probability is stochastic. In order to protect himself from a larger-than-expected survival probability, the provider can enter into an S-forward contract paying a fixed amount $K \in (0,1)$ and receiving an amount equal to the realised survival probability $\exp\{-\int_0^T \mu_x(v)\,dv\}$ at time T. In doing so, the survival probability that the provider is exposed to is certain, and corresponds to some fixed value K. If the contract is priced in such a way that there is no upfront cost at the inception, it must hold that

$$B(0,T)\,E_0^{\mathbb{Q}}\left(e^{-\int_0^T \mu_x(v)\,dv} - K(T)\right) = 0 \tag{3.8}$$

under the risk-adjusted measure $\mathbb{Q}$. Thus, the fixed amount can be identified to be the risk-adjusted survival probability, that is,

$$K(T) = E_0^{\mathbb{Q}}\left(e^{-\int_0^T \mu_x(v)\,dv}\right). \tag{3.9}$$

Assuming that there is a positive market price of longevity risk, the longevity risk hedger who pays the fixed leg and receives the floating leg bears the cost for entering an S-forward.[13] Following terminology in Biffis et al. (2014), the amount $K(T) = \tilde{S}_x(0,T)$ can be referred to as the swap rate of an S-forward with maturity T. In general, the mark-to-market price process $F(t)$ of an

[13] The risk-adjusted survival probability will be larger than the "best estimate" $\mathbb{P}$-survival probability if a positive market price of longevity risk is demanded, see Figure 4.

S-forward with fixed leg K (not necessarily $K(T)$ as in Eq. (3.9)) is given by

$$\begin{aligned} F(t) &= B(t,T)E_t^{\mathbb{Q}}\left(e^{-\int_0^T \mu_x(v)\,dv} - K\right) \\ &= B(t,T)E_t^{\mathbb{Q}}\left(e^{-\int_0^t \mu_x(v)\,dv} e^{-\int_t^T \mu_x(v)\,dv} - K\right) \\ &= B(t,T)\left(\bar{S}_x(0,t)\,\tilde{S}_{x+t}(t,T) - K\right) \end{aligned} \tag{3.10}$$

for $t \in [0,T]$. The quantity

$$\bar{S}_x(0,t) = e^{-\int_0^t \mu_x(v)\,dv}|_{\mathcal{F}_t} \tag{3.11}$$

is the realised survival probability, or the survivor index for the cohort, which is observable given $\mathcal{F}_t$.

The term $\bar{S}_x(0,t)\,\tilde{S}_{x+t}(t,T)$ that appears in Eq. (3.10) has a natural interpretation. Given information $\mathcal{F}_0$ at time $t = 0$, this term becomes $\tilde{S}_x(0,T)$, which is the risk-adjusted survival probability. As time moves on and more information $\mathcal{F}_t$, with $t \in (0,T)$, is revealed, the term $\bar{S}_x(0,t)\,\tilde{S}_{x+t}(t,T)$ is a product of the realised survival probability of the first t years, and the risk-adjusted survival probability in the next $(T-t)$ years. At maturity T, this product becomes the realised survival probability up to time T. In order words, one can think of $\bar{S}_x(0,t)\,\tilde{S}_{x+t}(t,T)$ as the T-year risk-adjusted survival probability with information known up to time t.

The price process $F(t)$ in Eq. (3.10) depends on the swap rate $\tilde{S}_{x+t}(t,T)$ of an S-forward written on the same cohort that is now aged $(x+t)$ at time t, with time to maturity $(T-t)$. If a liquid longevity market was developed, the swap rate $\tilde{S}_{x+t}(t,T)$ could be obtained from market data. As $\bar{S}_x(0,t)$ is observable at time t, the mark-to-market price process of an S-forward could be considered model-independent. However, since a longevity market is still in its development stage, market swap rates are not available and a model-based risk-adjusted survival probability $\tilde{S}_{x+t}(t,T)$ has to be used instead. An analytical formula for the mark-to-market price of an S-forward can be obtained if the risk-adjusted survival probability is expressed in a closed-form, which can be performed, for example, under the two-factor Gaussian mortality model.

Since a longevity swap is constructed as a portfolio of S-forwards, the price of a longevity swap is simply the sum of the individual S-forward prices.

3.3 Longevity Caps

A longevity cap, which is a portfolio of longevity caplets, provides a similar hedge to a longevity swap but is an option-type instrument. Consider again a scenario described in Section 3.2 where an annuity provider aims to hedge against larger-than-expected T-year survival probability of a particular cohort. Alternatively to hedging with an S-forward, the provider can enter into a long position of a longevity caplet with payoff at time T corresponding to

$$\max\left\{\left(e^{-\int_0^T \mu_x(v)\,dv} - K\right), 0\right\} \tag{3.12}$$

where $K \in (0,1)$ is the strike price.[14] If the realised survival probability is larger than K, the hedger receives an amount $\left(\exp\{-\int_0^T \mu_x(v)\,dv\} - K\right)$ from the longevity caplet. This payment can be regarded as a compensation for the increased payments that the provider has to make in the annuity portfolio, due to the larger-than-expected survival probability. There is no cash outflow if the realised survival probability is smaller than or equal to K. In other words, the

[14] The payoff of a longevity caplet is similar to the payoff of the option embedded in the principal-at-risk bond described in Biffis and Blake (2014).

longevity caplet allows the provider to "cap" its longevity exposure at K with no downside risk. Since a longevity caplet has a non-negative payoff, it comes at a cost. The price of a longevity caplet

$$C\ell(t;T,K) = B(t,T)E_t^{\mathbb{Q}}\left(\left(e^{-\int_0^T \mu_x(v)\,dv} - K\right)^+\right) \tag{3.13}$$

under the two-factor Gaussian mortality model is obtained in the following Proposition.

Proposition 3.1 *Under the two-factor Gaussian mortality model (Eq.* (2.1)*-Eq.* (2.3)*) the price at time t of a longevity caplet $C\ell(t;T,K)$, with maturity T and strike K, is given by*

$$C\ell(t;T,K) = \bar{S}_t\,\tilde{S}_t\,B(t,T)\,\Phi\left(\sqrt{\tilde{\Gamma}(t,T)} - d\right) - KB(t,T)\Phi\left(-d\right) \tag{3.14}$$

where $\bar{S}_t = \bar{S}_x(0,t)$ is the realised survival probability observable at time t, $\tilde{S}_t = \tilde{S}_{x+t}(t,T)$ is the risk-adjusted survival probability in the next $(T-t)$ years, $d = \frac{1}{\sqrt{\tilde{\Gamma}(t,T)}}\left(\ln\{K/(\bar{S}_t\tilde{S}_t)\} + \frac{1}{2}\tilde{\Gamma}(t,T)\right)$ and $\Phi(\cdot)$ denotes the cumulative distribution function of a standard Gaussian random variable.

Proof. Under the risk-adjusted measure $\mathbb{Q}$, we have, from Proposition (2.1), that

$$L \stackrel{\text{def}}{=} -\int_t^T \mu_x(v)dv \sim N(-\tilde{\Theta}(t,T), \tilde{\Gamma}(t,T)). \tag{3.15}$$

Using the simplified notation $\tilde{\Theta} = \tilde{\Theta}(t,T)$, $\tilde{\Gamma} = \tilde{\Gamma}(t,T)$ we can write

$$\begin{aligned}
C\ell(t;T,K) &= B(t,T)E_t^{\mathbb{Q}}\left((\bar{S}_t\,e^L - K)^+\right) \\
&= B(t,T)\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\tilde{\Gamma}}} e^{-\frac{1}{2}\left(\frac{\ell+\tilde{\Theta}}{\sqrt{\tilde{\Gamma}}}\right)^2}\left(\bar{S}_t\,e^{\ell} - K\right)^+ d\ell \\
&= B(t,T)\int_{\frac{\ln K/\bar{S}_t+\tilde{\Theta}}{\sqrt{\tilde{\Gamma}}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\ell^2}\left(\bar{S}_t\,e^{\ell\sqrt{\tilde{\Gamma}}-\tilde{\Theta}} - K\right) d\ell \\
&= B(t,T)\left(\bar{S}_t\,e^{\frac{1}{2}\tilde{\Gamma}-\tilde{\Theta}}\int_{\frac{\ln K/\bar{S}_t+\tilde{\Theta}}{\sqrt{\tilde{\Gamma}}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\ell-\sqrt{\tilde{\Gamma}}\right)^2} d\ell - K\int_{\frac{\ln K/\bar{S}_t+\Theta}{\sqrt{\tilde{\Gamma}}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\ell^2} d\ell\right).
\end{aligned}$$

Equation (3.14) follows using properties of $\Phi(\cdot)$ and noticing that $\tilde{S}_t = e^{\frac{1}{2}\tilde{\Gamma}-\tilde{\Theta}}$, that is, $\tilde{\Theta} = \frac{1}{2}\tilde{\Gamma} - \ln\tilde{S}_t$. □

Similar to an S-forward, the price of a longevity caplet depends on the product term $\bar{S}_x(0,t)\,\tilde{S}_{x+t}(t,T)$. In particular, a longevity caplet is said to be out-of-the-money if $K > \bar{S}_x(0,t)\,\tilde{S}_{x+t}(t,T)$; at-the-money if $K = \bar{S}_x(0,t)\,\tilde{S}_{x+t}(t,T)$; and in-the-money if $K < \bar{S}_x(0,t)\,\tilde{S}_{x+t}(t,T)$. Eq. (3.14), is verified using Monte Carlo simulation summarised in Table 2, where we set $r = 4\%$, $\lambda = 8.5$ and $t = 0$. Other parameters are as specified in Table 1.

Table 2
Pricing longevity caplet $C\ell(0;T,K)$ by the formula (Eq. (3.14)) and by Monte Carlo simulation of Eq. (3.13); [,] denotes the 95% confidence interval.

(T, K)	Exact	M.C. Simulation
(10, 0.6)	0.15632	0.15644 [0.15631, 0.15656]
(10, 0.7)	0.08929	0.08941 [0.08928, 0.08954]
(10, 0.8)	0.02261	0.02262 [0.02250, 0.02275]
(20, 0.3)	0.08373	0.08388 [0.08371, 0.08406]
(20, 0.4)	0.03890	0.03897 [0.03879, 0.03914]
(20, 0.5)	0.00525	0.00530 [0.00522, 0.00539]

Following the result of Proposition 3.1, the two-factor Gaussian mortality model leads to the price of a longevity caplet that is a function of the following variables:

- realised survival probability $\bar{S}_x(0,t)$ of the first t years;
- risk-adjusted survival probability $\tilde{S}_{x+t}(t,T)$ in the next $T-t$ years;
- interest rate r;
- strike price K;
- time to maturity $(T-t)$; and
- standard deviation $\sqrt{\tilde{\Gamma}(t,T)}$, which is a function of the time to maturity and the model parameters.

Since the quantity $\exp\left\{-\int_0^T \mu_x(v)\,dv\right\}$ is log-normally distributed under the two-factor Gaussian mortality model, Eq. (3.14) resembles the Black-Scholes formula for option pricing where the underlying stock price follows a geometric Brownian motion. In our setup, the stock price at time t is replaced by the T-year risk-adjusted survival probability $\bar{S}_x(0,t)\,S_{x+t}(t,T)$ with information available up to time t. While the stock is traded and can be modelled directly using market data, the underlying of a longevity caplet is the survival probability which is not tradable but can be determined as an output from the dynamics of mortality intensity. As a result, the role of the stock price volatility in the Black-Scholes formula is played by the standard deviation of the integral of the mortality intensity $\int_t^T \mu_x(v)\,dv$. Since the integral $\int_t^T \mu_x(v)\,dv$ captures the whole history of the mortality intensity $\mu_x(t)$ from t to T under $\mathbb{Q}$, one can interpret the standard deviation $\sqrt{\tilde{\Gamma}(t,T)}$ as the volatility of the risk-adjusted aggregated longevity risk of a cohort aged $x+t$ at time t, for the period from t to T.

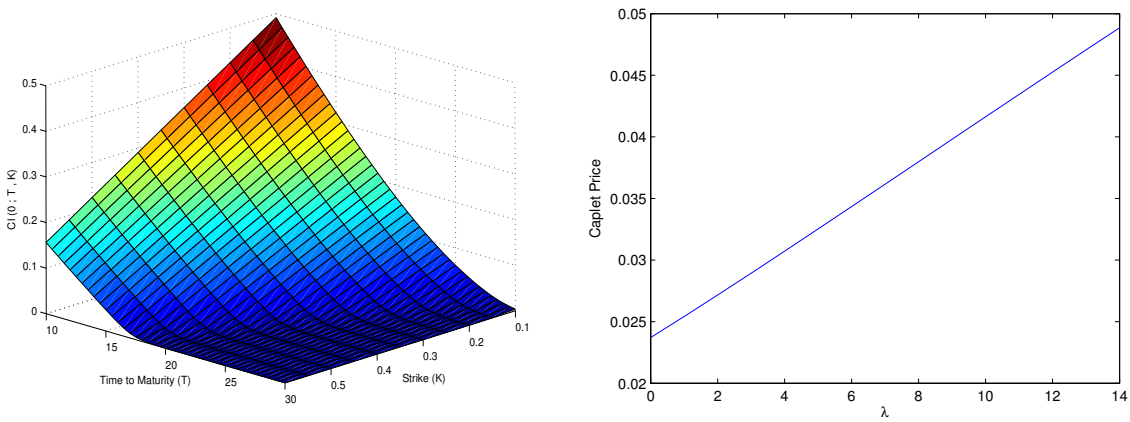

Fig. 5. Caplet price as a function of (left panel) T and K and (right panel) λ where $K = 0.4$ and $T = 20$.

The left panel of Figure 5 shows caplet prices for a cohort aged $x = 65$, using parameters as specified in Table 1, as a function of time to maturity T and strike K. We set $r = 0.04$, $\lambda = 8.5$ and $t = 0$ such that $\bar{S}_x(0,0) = 1$. A lower strike price indicates that the buyer of a caplet is willing to pay more to secure a better protection against a larger-than-expected survival probability. On the other hand, when the time to maturity T is increasing, the underlying survival probability is likely to take smaller values, which leads to a higher probability for the caplet to become out-of-the-money at maturity for a fixed K, see Eq. (3.12). Consequently, for a fixed K the caplet price decreases with increasing T.

The right panel of Figure 5 illustrates the effect of the market price of longevity risk λ on the caplet price. The price of a caplet increases with increasing λ. As shown in Figure 4, a larger value of λ will lead to an improvement in survival probability under $\mathbb{Q}$. Thus, a higher caplet price is observed since the underlying survival probability is larger (on average) under $\mathbb{Q}$ when λ increases, see Eq. (3.13).

Since longevity cap is constructed as a portfolio of longevity caplets, it can be priced as a sum of individual caplet prices, see also Section 4.1.2.

4 Managing Longevity Risk in a Hypothetical Life Annuity Portfolio

Hedging features of a longevity swap and cap are examined for a hypothetical life annuity portfolio subject to longevity risk. Factors considered include the market price of longevity risk, the term to maturity of hedging instruments and the size of the underlying annuity portfolio.

4.1 Setup

We consider a hypothetical life annuity portfolio that consists of a cohort aged $x = 65$. The size of the portfolio that corresponds to the number of policyholders, is denoted by n. The underlying mortality intensity for the cohort follows the two-factor Gaussian mortality model described in Section 2, and the model parameters are specified in Table 1. We assume that there is no loading for the annuity policy and expenses are not included.

Further, we assume a single premium, whole life annuity of \$1 per year payable in arrears conditional on the survival of the annuitant to the payment dates. The fair value, or the premium, of the annuity evaluated at $t = 0$ is given by

$$a_x = \sum_{T=1}^{\omega-x} B(0,T)\,\tilde{S}_x(0,T) \tag{4.1}$$

where $r = 4\%$ and $\omega = 110$ is the maximum age allowed in the mortality model. The life annuity provider, thus, receives a total premium, denoted by A, for the whole portfolio corresponding to the sum of individual premiums:

$$A = n\,a_x. \tag{4.2}$$

This is the present value of the asset held by the annuity provider at $t = 0$. Since the promised annuity cashflows depend on the death times of annuitants in the portfolio, the present value of the liability is subject to randomness caused by the stochastic dynamics of the mortality intensity. The present value of the liability for each policyholder, denoted by L_k, is determined by the death time τ_k of the policyholder, and is given by

$$L_k = \sum_{T=1}^{\lfloor \tau_k \rfloor} B(0,T) \tag{4.3}$$

for a simulated τ_k, with $\lfloor q \rfloor$ denoting the next smaller integer of a real number q. The present value of the liability L for the whole portfolio is obtained as a sum of individual liabilities:

$$L = \sum_{k=1}^{n} L_k. \tag{4.4}$$

The algorithm for simulating death times of annuitants, which requires a single simulated path for the mortality intensity of the cohort, is summarised in Appendix A. The discounted surplus distribution (D_{no}) of an unhedged annuity portfolio is obtained by setting

$$D_{\text{no}} = A - L. \tag{4.5}$$

The impact of longevity risk is captured by simulating the discounted surplus distribution where each sample is determined by the realised mortality intensity of a cohort. Since traditional pricing

and risk management of life annuity relies on diversification effect, or the law of large numbers, we consider the discounted surplus distribution per policy

$$D_{\text{no}}/n. \tag{4.6}$$

Figure 6 shows the discounted surplus distribution per policy without longevity risk (i.e. when setting $\sigma_1 = \sigma = 0$) with different portfolio sizes, varying from $n = 2000$ to 8000. As expected, the mean of the distribution is centred around zero as there is no loading assumed in the pricing algorithm, while the standard deviation diminishes as the number of policies increases.

In the following we consider a longevity swap and a cap as hedging instruments. These are index-based instruments where the payoffs depend on the survivor index, or the realised survival probability (Eq. (3.11)), which is in turn determined by the realised mortality intensity. We do not consider basis risk[15] but due to a finite portfolio size, the actual proportion of survivors, $\frac{n-N_t}{n}$, where N_t denotes the number of deaths experienced by a cohort during the period $[0, t]$, will be in general similar, but not identical, to the survivor index (Appendix A). As a result, the static hedge will be able to reduce systematic mortality risk, whereas the idiosyncratic mortality risk component will be retained by the annuity provider.

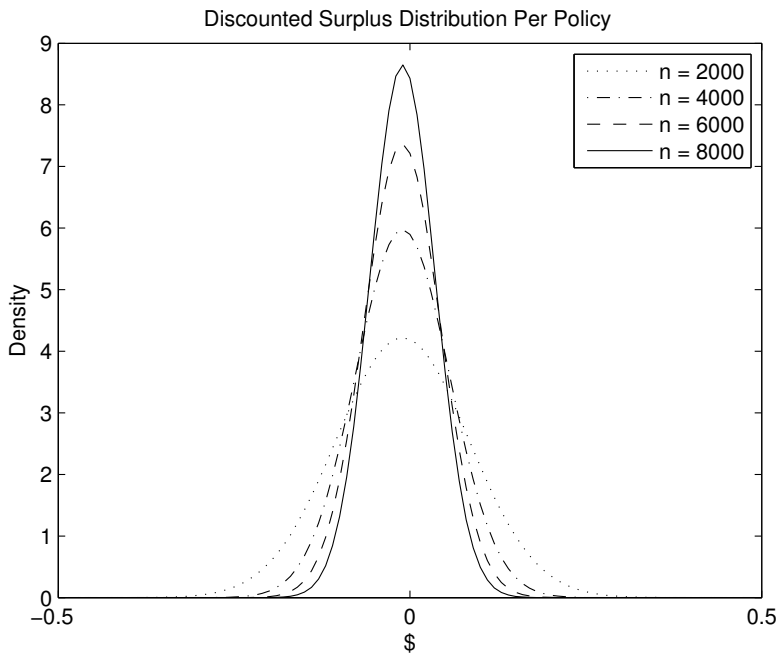

Fig. 6. Discounted surplus distribution per policy without longevity risk with different portfolio size (n).

4.1.1 A Swap-Hedged Annuity Portfolio

For an annuity portfolio hedged by an index-based longevity swap, payments from the swap

$$n\left(e^{-\int_0^T \mu_x(v)\,dv} - K(T)\right) \tag{4.7}$$

at time $T \in \{1, ..., \hat{T}\}$ depend on the realised mortality intensity, where $\hat{T}$ denotes the term to maturity of the longevity swap. The number of policyholders n acts as the notional amount of the swap contract so that the quantity $n \exp\{-\int_0^T \mu_x(v)\,dv\}$ represents the number of survivors

[15] If basis risk is present, we need to distinguish between the mortality intensity for the population (μ_x^I) and mortality intensity for the cohort (μ_x) underlying the annuity portfolio, see Biffis et al. (2014).

implied by the realised mortality intensity at time T. We fix the strike of a swap to the risk-adjusted survival probability, that is,

$$K(T) = \tilde{S}_x(0,T) = E_0^{\mathbb{Q}}\left(e^{-\int_0^T \mu_x(v)\,dv}\right) \tag{4.8}$$

such that the price of a swap is zero at $t = 0$, see Section 3.2. The discounted surplus distribution of a swap-hedged annuity portfolio can be expressed as

$$D_{\text{swap}} = A - L + F_{\text{swap}} \tag{4.9}$$

where

$$F_{\text{swap}} = n \sum_{T=1}^{\hat{T}} B(0,T)\left(e^{-\int_0^T \mu_x(v)\,dv} - \tilde{S}_x(0,T)\right) \tag{4.10}$$

is the (random) discounted cashflow coming from a long position in the longevity swap. The discounted surplus distribution per policy of a swap-hedged annuity portfolio is determined by D_{swap}/n.

4.1.2 A Cap-Hedged Annuity Portfolio

For an annuity portfolio hedged by an index-based longevity cap, the cashflows

$$n \max\left\{\left(e^{-\int_0^T \mu_x(v)\,dv} - K(T)\right), 0\right\} \tag{4.11}$$

at $T \in \{1, ..., \hat{T}\}$ are payments from a long position in the longevity cap. We set

$$K(T) = S_x(0,T) = E_0^{\mathbb{P}}\left(e^{-\int_0^T \mu_x(v)\,dv}\right) \tag{4.12}$$

such that the strike for a longevity caplet is the "best estimated" survival probability given $\mathcal{F}_0$. [16] The discounted surplus distribution of a cap-hedged annuity portfolio is given by

$$D_{\text{cap}} = A - L + F_{\text{cap}} - C_{\text{cap}} \tag{4.13}$$

where

$$F_{\text{cap}} = n \sum_{T=1}^{\hat{T}} B(0,T) \max\left\{\left(e^{-\int_0^T \mu_x(v)\,dv} - S_x(0,T)\right), 0\right\} \tag{4.14}$$

is the (random) discounted cashflow from holding the longevity cap and

$$C_{\text{cap}} = n \sum_{T=1}^{\hat{T}} C\ell\left(0; T, S_x(0,T)\right) \tag{4.15}$$

is the price of the longevity cap. The discounted surplus distribution per policy of a cap-hedged annuity portfolio is given by D_{cap}/n.

4.2 Results

Hedging results are summarised by means of summary statistics that include mean, standard deviation (std. dev.), skewness, as well as Value-at-Risk (VaR) and Expected Shortfall (ES) of

[16] For a longevity swap, the risk-adjusted survival probability is used as a strike price so that the price of a longevity swap is zero at inception. In contrast, a longevity cap has non-zero price and $S_x(0,T)$ is the most natural choice for a strike.

the discounted surplus distribution per policy of an unhedged, a swap-hedged and a cap-hedged annuity portfolio. Skewness is included since the payoff of a longevity cap is nonlinear and the resulting distribution of a cap-hedged annuity portfolio is not symmetric. VaR is defined as the q-quantile of the discounted surplus distribution per policy. ES is defined as the expected loss of the discounted surplus distribution per policy given the loss is at or below the q-quantile. We fix $q = 0.01$ so that the confidence interval for VaR and ES corresponds to 99%. We use 5,000 simulations to obtain the distribution for the discounted surplus. Hedge effectiveness is examined with respect to (w.r.t.) different assumptions underlying the market price of longevity risk (λ), the term to maturity of hedging instruments ($\hat{T}$) and the portfolio size (n). Parameters for the base case are as specified in Table 3.

Table 3
Parameters for the base case.

λ	$\hat{T}$ (years)	n
8.5	30	4000

4.2.1 Hedging Features w.r.t. Market Price of Longevity Risk

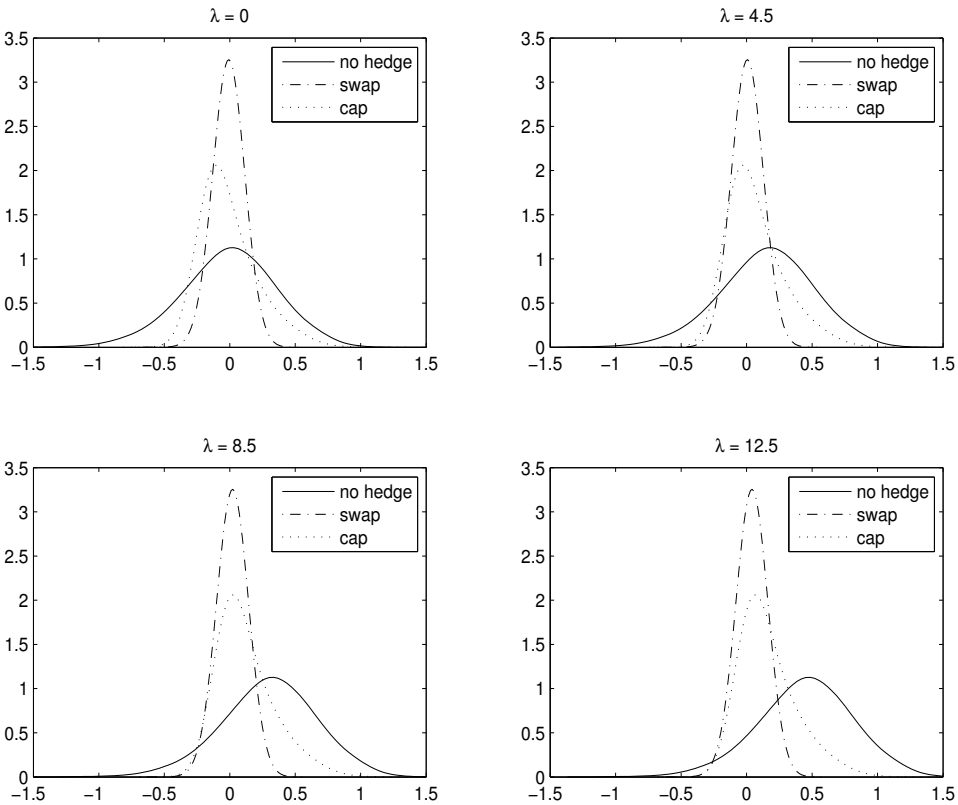

Fig. 7. Effect of the market price of longevity risk λ on the discounted surplus distribution per policy.

The market price of longevity risk λ is one of the factors that determines prices of longevity derivatives and life annuity policies. Since payoffs of a longevity swap, a cap and a life annuity are contingent on the same underlying mortality intensity of a cohort, all these products are priced using the same λ. Figure 7 and Table 4 illustrate the effect of changing λ on the distributions of an unhedged, a swap-hedged and a cap-hedged annuity portfolio. The degree of longevity risk can be quantified by the standard deviation, the VaR and the ES of the distributions. We observe that increasing λ leads to the shift of the distribution to the right, resulting in a higher average surplus. On the other hand, changing λ has no impact on the standard deviation and the skewness of the distribution.

Table 4
Hedging features of a longevity swap and cap w.r.t. market price of longevity risk λ.

	Mean	Std.dev.	Skewness	$\text{VaR}_{0.99}$	$\text{ES}_{0.99}$
			$\lambda = 0$		
No hedge	-0.0076	0.3592	-0.2804	-0.9202	-1.1027
Swap-hedged	-0.0089	0.0718	-0.1919	-0.1840	-0.2231
Cap-hedged	-0.0086	0.2054	1.0855	-0.3193	-0.3515
			$\lambda = 4.5$		
No hedge	0.1520	0.3592	-0.2804	-0.7606	-0.9431
Swap-hedged	0.0048	0.0718	-0.1919	-0.1703	-0.2094
Cap-hedged	0.0682	0.2054	1.0855	-0.2425	-0.2746
			$\lambda = 8.5$		
No hedge	0.2978	0.3592	-0.2804	-0.6148	-0.7973
Swap-hedged	0.0204	0.0718	-0.1919	-0.1547	-0.1938
Cap-hedged	0.1205	0.2054	1.0855	-0.1903	-0.2224
			$\lambda = 12.5$		
No hedge	0.4475	0.3592	-0.2804	-0.4650	-0.6476
Swap-hedged	0.0398	0.0718	-0.1919	-0.1354	-0.1744
Cap-hedged	0.1619	0.2054	1.0855	-0.1489	-0.1810

For an unhedged annuity portfolio, a higher λ leads to higher premium for the life annuity policy since the annuity price is determined by the risk-adjusted survival probability $\tilde{S}_x(0,T)$, see Eq. (4.1). In other words, an increase in the annuity price compensates the provider for the longevity risk undertaken when selling life annuity policies. There is also a trade-off between risk premium and affordability. Setting a higher premium will clearly improve the risk and return of an annuity business, it might, however, reduce the interest of potential policyholders. An empirical relationship between implied longevity and annuity prices is studied in Chigodaev et al. (2014).

When life annuity portfolio is hedged using a longevity swap, the standard deviation and the absolute values of the VaR and the ES reduce substantially. The higher return obtained by charging a larger market price of longevity risk in life annuity policies is offset by an increased price paid implicitly in the swap contract (since $\tilde{S}_x(0,T) \geq S_x(0,T)$ in Eq. (4.10)). It turns out that as λ increases an extra return earned in the annuity portfolio and the higher implicit cost of the longevity swap nearly offset each other out on average. The net effect is that a swap-hedged annuity portfolio remains to a great extent unaffected by the assumption on λ, leading only to a very minor increase in the mean of the distribution.

For a cap-hedged annuity portfolio, the discounted surplus distribution is positively skewed since a longevity cap allows an annuity provider to get exposure to the upside potential when policyholders live shorter than expected. Compared to an unhedged portfolio, the standard deviation and the absolute values of the VaR and the ES are also reduced but the reduction is smaller compared to a swap-hedged portfolio. When λ increases, we observe that the mean of the distribution for a cap-hedged portfolio increases faster than for a swap-hedged portfolio but slower than for an unhedged portfolio. It can be explained by noticing that when the survival probability of a cohort is overestimated, that is, when annuitants turn out to live shorter than expected, holding a longevity cap has no effect (besides paying the price of a cap for longevity protection at the inception of the contract) while there is a cash outflow when holding a longevity swap, see Eq. (4.10) and Eq. (4.14).

In the longevity risk literature, the VaR and the ES are of a particular importance as they are the main factors determining the capital reserve when dealing with exposure to longevity risk (Meyricke and Sherris (2014)). As shown in Table 4, the difference between a swap-hedged and a cap-hedged portfolio in terms of the VaR and the ES becomes smaller when λ increases. In fact, for $\lambda \geq 17.5$, a longevity cap becomes more effective in reducing the tail risk of an annuity portfolio compared to a longevity swap. [17] This result suggests that a longevity cap, besides being able to capture the upside potential, can be a more effective hedging instrument than a longevity swap in terms of reducing the VaR and the ES when the demanded market price of longevity risk

[17] Given $\lambda = 17.5$, the VaR and the ES for a swap-hedged portfolio are -0.1051 and -0.1441 respectively. For a cap-hedged portfolio they become -0.1038 and -0.1360, respectively.

λ is large.

4.2.2 Hedging Features w.r.t. Term to Maturity

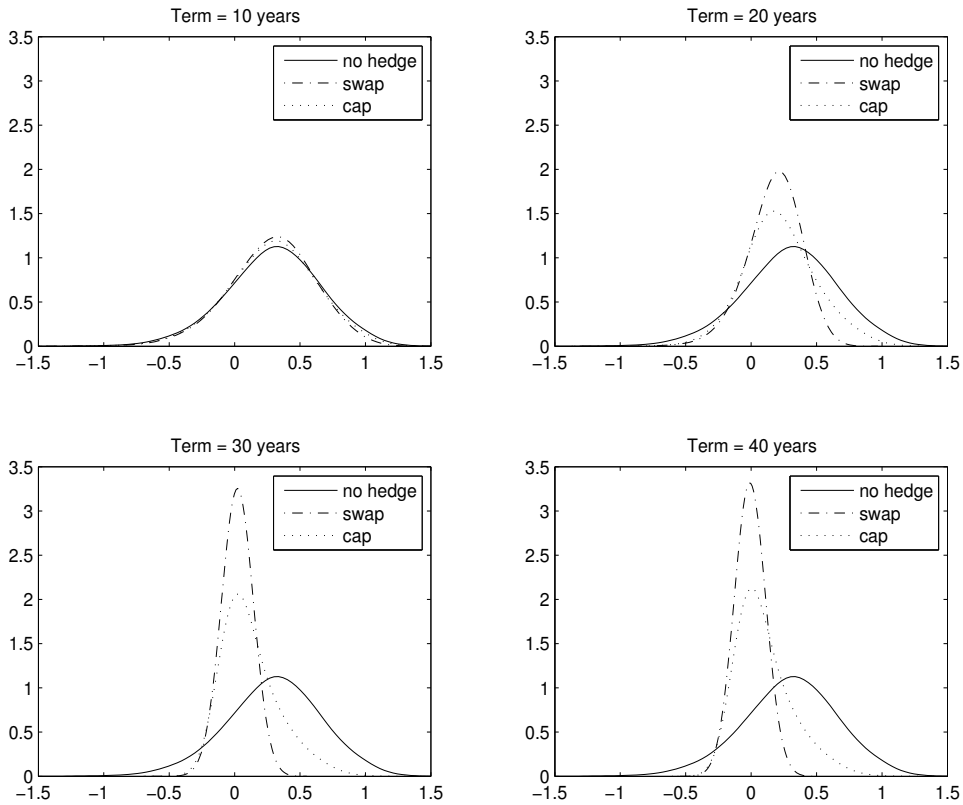

Fig. 8. Effect of the term to maturity $\hat{T}$ of the hedging instruments on the discounted surplus distribution per policy.

Table 5
Hedging features of a longevity swap and cap w.r.t. term to maturity $\hat{T}$.

	Mean	Std.dev.	Skewness	$\text{VaR}_{0.99}$	$\text{ES}_{0.99}$
			$\hat{T} = 10$ Years		
No hedge	0.2978	0.3592	-0.2804	-0.6148	-0.7973
Swap-hedged	0.2820	0.2911	-0.3871	-0.5707	-0.7490
Cap-hedged	0.2893	0.2989	-0.2661	-0.5801	-0.7592
			$\hat{T} = 20$ Years		
No hedge	0.2978	0.3592	-0.2804	-0.6148	-0.7973
Swap-hedged	0.1740	0.1794	-0.7507	-0.3656	-0.5061
Cap-hedged	0.2234	0.2310	0.2006	-0.3870	-0.5259
			$\hat{T} = 30$ Years		
No hedge	0.2978	0.3592	-0.2804	-0.6148	-0.7973
Swap-hedged	0.0204	0.0718	-0.1919	-0.1547	-0.1938
Cap-hedged	0.1205	0.2054	1.0855	-0.1903	-0.2224
			$\hat{T} = 40$ Years		
No hedge	0.2978	0.3592	-0.2804	-0.6148	-0.7973
Swap-hedged	-0.0091	0.0668	0.0277	-0.1616	-0.1869
Cap-hedged	0.0984	0.1999	1.1527	-0.1909	-0.2131

Table 5 and Figure 8 summarize hedging results with respect to the term to maturity of hedging instruments. Due to the long-term nature of the contracts, the hedges are ineffective for $\hat{T} \leq 10$ years and the standard deviations are reduced only by around $17 - 19\%$ for both instruments. The lower left panel of Figure 3 shows that there is little randomness around the realised survival probability for the first few years for a cohort aged 65, and consequently the hedges are

insignificant when $\hat{T}$ is short.

The difference in hedge effectiveness between $\hat{T} = 30$ and $\hat{T} = 40$ for both instruments is also insignificant. In fact, the longevity risk underlying the annuity portfolio becomes small after 30 years since the majority of annuitants has already deceased before reaching the age of 95. In our model setup the chance for a 65 years old to live up to 95 is around 6% (Figure 4 with $\lambda = 0$) and, hence, only around $4000 \times 6\% = 240$ policies will still be in-force after 30 years. Much of the risk left is attributed to idiosyncratic mortality risk, and hedging longevity risk for a small portfolio using index-based instruments is of limited use.

For a swap-hedged portfolio, the standard deviation is reduced significantly when $\hat{T} > 20$ years. The mean surplus, on the other hand, drops to nearly zero since there is a higher cost implied for the hedge with increasing number of S-forwards involved to form the swap as $\hat{T}$ increases.

Similar hedging features with respect to $\hat{T}$ are observed for a longevity cap. However, the skewness of the distribution of a cap-hedged portfolio increases with increasing $\hat{T}$. It can be explained by noticing that while a longevity cap is able to capture the upside potential regardless of $\hat{T}$, it provides a better longevity risk protection when $\hat{T}$ is larger. As a result, the distribution of a cap-hedged portfolio becomes more asymmetric when $\hat{T}$ increases.

4.2.3 Hedging Features w.r.t. Portfolio Size

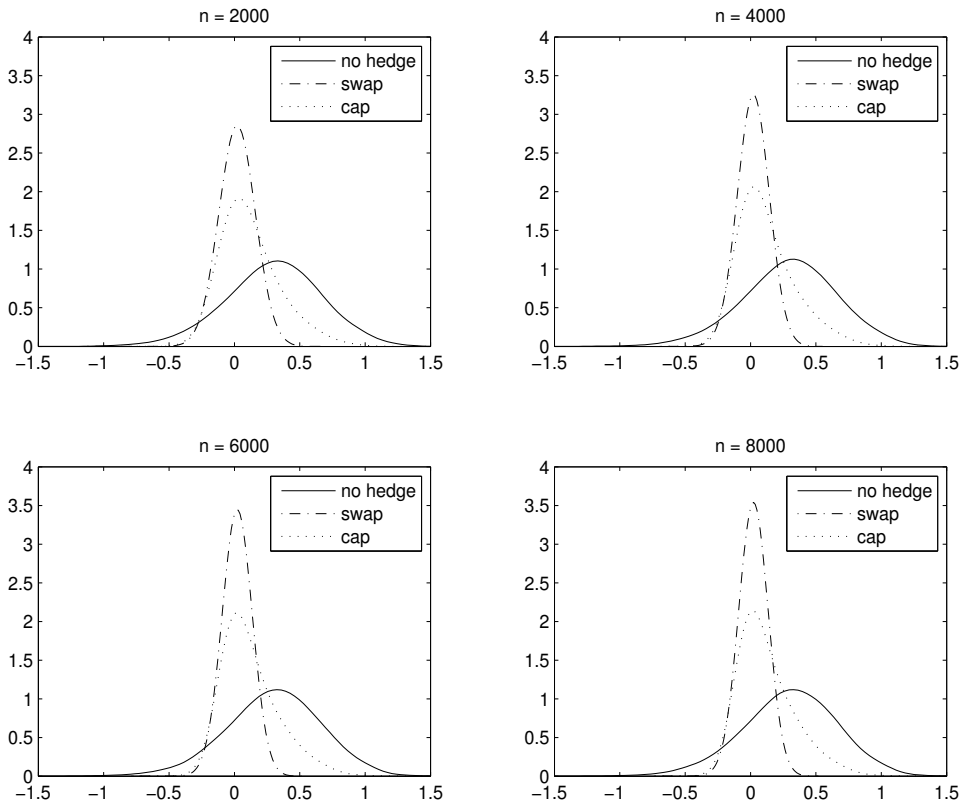

Fig. 9. Effect of the portfolio size n on the discounted surplus distribution per policy.

Table 6 and Figure 9 demonstrate hedging features of a longevity swap and a cap with changing portfolio size n. We observe a decrease in standard deviation, as well as the VaR and the ES (in absolute terms) when portfolio size increases. Compared to an unhedged portfolio, the reduction in the standard deviation and the risk measures is larger for a swap-hedged portfolio, compared

Table 6
Hedging features of a longevity swap and cap w.r.t. different portfolio size (n).

	Mean	Std.dev.	Skewness	$\text{VaR}_{0.99}$	$\text{ES}_{0.99}$
			$n = 2000$		
No hedge	0.2973	0.3646	-0.2662	-0.6360	-0.8107
Swap-hedged	0.0200	0.0990	-0.1615	-0.2120	-0.2653
Cap-hedged	0.1200	0.2160	0.9220	-0.2432	-0.2944
			$n = 4000$		
No hedge	0.2978	0.3592	-0.2804	-0.6148	-0.7973
Swap-hedged	0.0204	0.0718	-0.1919	-0.1547	-0.1938
Cap-hedged	0.1205	0.2054	1.0855	-0.1903	-0.2224
			$n = 6000$		
No hedge	0.2977	0.3566	-0.2786	-0.6363	-0.8001
Swap-hedged	0.0204	0.0594	-0.3346	-0.1259	-0.1660
Cap-hedged	0.1204	0.2016	1.1519	-0.1639	-0.2051
			$n = 8000$		
No hedge	0.2982	0.3554	-0.2920	-0.6060	-0.7876
Swap-hedged	0.0209	0.0536	-0.5056	-0.1190	-0.1595
Cap-hedged	0.1209	0.1992	1.1616	-0.1598	-0.1991

to a cap-hedged portfolio. Recall that idiosyncratic mortality risk becomes significant when n is small. We quantify the effect of the portfolio size on hedge effectiveness by introducing the measure of longevity risk reduction R, defined in terms of the variance of the discounted surplus per policy, that is,

$$R = 1 - \frac{\text{Var}(\bar{D}^*)}{\text{Var}(\bar{D})}, \tag{4.16}$$

where $\text{Var}(\bar{D}^*)$ and $\text{Var}(\bar{D})$ represent the variances of the discounted surplus distribution per policy for a hedged and an unhedged annuity portfolio, respectively. The results are reported in Table 7.

Table 7
Longevity risk reduction R of a longevity swap and cap w.r.t. different portfolio size (n).

n	2000	4000	6000	8000
R_{swap}	92.6%	96.0%	97.2%	97.7%
R_{cap}	64.9%	67.3%	68.0%	68.6%

Li and Hardy (2011) consider hedging longevity risk using a portfolio of q-forwards and find the longevity risk reduction of 77.6% and 69.6% for portfolio size of 10,000 and 3,000, respectively. In contrast to Li and Hardy (2011), we do not consider basis risk and the result of using longevity swap as a hedging instrument leads to a greater risk reduction. Overall, our results indicate that hedge effectiveness for an index-based longevity swap and a cap diminishes with decreasing n since idiosyncratic mortality risk cannot be effectively diversified away for a small portfolio size. Even though a longevity cap is less effective in reducing the variance, part of the dispersion is attributed to its ability of capturing the upside of the distribution when survival probability of a cohort is overestimated. From Table 6 we also observe that the distribution becomes more positively skewed for a cap-hedged portfolio when n increases, which is a consequence of having a larger exposure to longevity risk with increasing number of policyholders in the portfolio.

5 Conclusion

Life and pension annuities are the most important types of post-retirement products offered by annuity providers to help securing lifelong incomes for the rising number of retirees. While interest rate risk can be managed effectively in the financial markets, longevity risk is a major concern for annuity providers as there are only limited choices available to mitigate the long-term risk. Development of effective financial instruments for longevity risk in capital markets is arguably the best solution available.

Two types of longevity derivatives, a longevity swap and a cap, are analysed in this paper from a pricing and hedging perspective. We apply a tractable Gaussian mortality model to capture the longevity risk, and derive explicit formulas for important quantities such as survival probabilities and prices of longevity derivatives. Hedge effectiveness and features of an index-based longevity swap and a cap used as hedging instruments are examined using a hypothetical life annuity portfolio exposed to longevity risk.

Our results suggest that the market price of longevity risk λ is a small contributor to hedge effectiveness of a longevity swap since a higher annuity price is partially offset by an increased cost of hedging when λ is taken into account. It is shown that a longevity cap, while being able to capture the upside potential when survival probabilities are overestimated, can be more effective in reducing longevity tail risk compared to a longevity swap, provided that λ is large enough. The term to maturity $\hat{T}$ is an important factor in determining hedge effectiveness. However, the difference in hedge effectiveness is only marginal when $\hat{T}$ increases from 30 to 40 years for an annuity portfolio consisting of a single cohort aged 65 initially. This is due to the fact that only a small number of policies will still be in-force after a long period of time (30 to 40 years), and index-based instruments turn out to be ineffective when idiosyncratic mortality risk becomes a larger contributor to the overall risk, compared to systematic mortality risk. The effect of the portfolio size n on hedge effectiveness is quantified and compared with the result obtained in Li and Hardy (2011) where population basis risk is taken into account. In addition, we find that the skewness of the surplus distribution of a cap-hedged portfolio is sensitive to the term to maturity and the portfolio size, and, as a result, the difference between a longevity swap and a cap when used as hedging instruments becomes more pronounced for larger $\hat{T}$ and n.

As discussed in Biffis and Blake (2014), developing a liquid longevity market requires reliable and well-designed financial instruments that can attract sufficient amount of interests from both buyers and sellers. Besides of a longevity swap, which is so far a common longevity hedging choice for annuity providers, option-type instruments such as longevity caps can provide hedging features that linear products cannot offer. A longevity cap is shown to have alternative hedging properties compared to a swap, and this option-type instrument would also appeal to certain classes of investors interested in receiving premiums by selling a longevity insurance. Further research on the design of longevity-linked instruments from the perspectives of buyers and sellers would provide a further step towards the development of an active longevity market.

A Appendix

To simulate death times of annuitants, we notice that once a sample of the mortality intensity is obtained, the Cox process becomes an inhomogeneous Poisson process and the first jump times, which are interpreted as death times, can be simulated as follows (see e.g. Brigo and Mercurio (2007)):

(1) Simulate the mortality intensity $\mu_x(t)$ from $t = 0$ to $t = \omega - x$.
(2) Generate a standard exponential random variable ξ. For example, using an inverse transform method, we have $\xi = -\ln(1-u)$ where $u \sim \text{Uniform}(0,1)$.
(3) Set the death time τ to be the smallest T such that $\xi \leq \int_0^T \mu_x(s)\,ds$. If $\xi > \int_0^{\omega-x} \mu_x(s)\,ds$ then set $\tau = \omega - x$.
(4) Repeat step (2) and (3) to obtain another death time.

The payoff of an index-based hedging instrument depends on the realised survival probability $\exp\{-\int_0^t \mu_x(v)\,dv\}$. The payoff of a customised instrument, on the other hand, depends on the proportion of survivors, $\frac{n-N_t}{n}$, underlying an annuity portfolio where the number of deaths, N_t,

is obtained by counting the number of simulated death times that are smaller than t. Note that

$$e^{-\int_0^t \mu_x(v)\,dv} \approx \frac{n - N_t}{n} \tag{A.1}$$

and the accuracy of the approximation improves when n increases.

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